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Extremely primitive classical groups

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ABSTRACT

A primitive permutation group is said to be extremely primitive if it is not regular and a point stabilizer acts primitively on each of its orbits. By a theorem of Mann and the second and third authors, every finite extremely primitive group is either almost simple or of affine type. In this paper, we determine the examples in the case of almost simple classical groups. They comprise the 2-transitive actions of $\mathrm{PSL}_2(q)$ and its extensions of degree $q+1$, and of $\mathrm{Sp}_{2m}(2)$ of degrees $2^{2m-1} \pm 2^{m-1}$, together with the 3/2-transitive actions of $\mathrm{PSL}_2(q)$ on cosets of D_{q+1}, with $q+1$ a Fermat prime. In addition to these three families, there are four individual examples.

1. Introduction

A non-regular primitive permutation group G on a set Ω is said to be *extremely primitive* if a point stabilizer $H = G_\alpha$ acts primitively on each of its orbits. Equivalently, G is extremely primitive if $H \cap H^x$ is a maximal subgroup of H for all $x \in G \setminus H$. Moreover, by an old theorem of Manning [18], if G is extremely primitive on Ω then G_α is faithful on each of its orbits in $\Omega \setminus \{\alpha\}$, so $H \cap H^x$ is also core-free in H. For example, every 2-primitive group G on Ω is extremely primitive, and the finite groups with this property can be determined via the classification of finite simple groups.

By a theorem of Mann et al. [17, Theorem 1.1], every finite extremely primitive group is either almost simple or of affine type, and the affine examples are known up to a finite number of possibilities. The purpose of this paper is to determine the examples in the case of almost simple classical groups. Our main theorem is the following.

Theorem 1.1. *Let G be a finite almost simple classical primitive permutation group, with point stabilizer H and socle G_0. Then G is extremely primitive if and only if (G, H) is one of the cases listed in Table 1.*

Remark 1.2. In Table 1, the *type of H* describes the approximate group-theoretic structure of H; this is consistent with the notation used in [14]. In the first row, P_1 denotes a Borel subgroup of G, which is the stabilizer of a 1-dimensional subspace of the natural G_0-module. In the third row we require $q+1$ to be a *Fermat prime*, so $q = 2^{2^r}$ for some positive integer r. The table contains each example up to permutational isomorphism (but with the case $G_0 \cong A_6$ of degree 10 occurring in both line 1 and line 2). Note that we are not claiming that every group of the given shape in rows 5 and 6 provides an extremely primitive example—we refer the reader to the specific proposition recorded in the final column of the table for the precise details.

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Table 1
The extremely primitive classical groups.

Line	G_0	Type of H	Conditions	Reference
1	$\mathrm{PSL}_2(q)$	P_1	$q \geq 4$	3.6, 4.2, 5.3 and 8.1
2	$\mathrm{PSp}_n(2)'$	$O_n^{\epsilon}(2)$	$n \geq 4$	9.4, 4.2 and 5.4
3	$\mathrm{PSL}_2(q)$	$\mathrm{GL}_1(q^2)$	$G = G_0$, $q > 2$, $q+1$ Fermat	5.3
4	$\mathrm{PSL}_4(2)$	A_7		10.4
5	$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_3(4)$	$G = G_0.2^2$ or $G = G_0.2$	10.4
6	$\mathrm{PSL}_3(4)$	A_6	$G = G_0.2^2$ or $G = G_0.2$	10.4
7	$\mathrm{PSL}_2(11)$	A_5	$G = G_0$	10.4

Table 2
The $\mathcal{C}_i$ families.

$\mathcal{C}_1$	Stabilizers of subspaces, or pairs of subspaces, of V
$\mathcal{C}_2$	Stabilizers of decompositions $V = \bigoplus_{i=1}^{t} V_i$, where $\dim V_i = a$
$\mathcal{C}_3$	Stabilizers of prime index extension fields of $\mathbb{F}_q$
$\mathcal{C}_4$	Stabilizers of decompositions $V = V_1 \otimes V_2$
$\mathcal{C}_5$	Stabilizers of prime index subfields of $\mathbb{F}_q$
$\mathcal{C}_6$	Normalizers of symplectic-type r-groups
$\mathcal{C}_7$	Stabilizers of decompositions $V = \bigotimes_{i=1}^{t} V_i$, where $\dim V_i = a$
$\mathcal{C}_8$	Stabilizers of non-degenerate forms on V
$\mathcal{C}_9$	Almost simple irreducible subgroups of G
$\mathcal{C}_{10}$	Novelty subgroups ($G_0 = \mathrm{P}\Omega_8^+(q)$ or $\mathrm{Sp}_4(q)'$ (q even), only)

Remark 1.3. A classification of the almost simple extremely primitive groups with a sporadic or alternating socle is forthcoming in [6], and the extremely primitive groups of exceptional Lie type will also be the subject of a future paper.

The proof of Theorem 1.1 requires a detailed analysis of the maximal subgroups of finite classical groups. Let G be an almost simple classical group over $\mathbb{F}_q$ with socle G_0 and natural module V, where $q = p^f$ and p is a prime. The main theorem on the subgroup structure of classical groups is due to Aschbacher. In [1], eight collections of subgroups of G are defined, labelled $\mathcal{C}_i$ for $1 \leq i \leq 8$, and it is shown that if H is a maximal subgroup of G such that $G = G_0H$ then either H is contained in one of these natural subgroup collections, or it belongs to a family of almost simple subgroups which act irreducibly on V (we use $\mathcal{C}_9$ to denote this latter collection). A small additional collection of maximal subgroups (denoted by $\mathcal{C}_{10}$) arises when $G_0 = \mathrm{P}\Omega_8^+(q)$ or $\mathrm{Sp}_4(q)'$ (q even), due to the existence of certain exceptional outer automorphisms (see Section 11). See Table 2 for a rough description of the $\mathcal{C}_i$ families. A detailed analysis of the subgroups in the $\mathcal{C}_i$ collections with $1 \leq i \leq 8$ is given by Kleidman and Liebeck [14], and throughout this paper we adopt the notation therein.

In the forthcoming paper [5], Guralnick, Saxl and the first author determine the pairs (G, H), where G is a classical group as before, H is a maximal subgroup of G and $H \cap H^x = 1$ for some $x \in G$. In the language of permutation groups, this provides a classification of the primitive almost simple classical groups with a base of size 2 (here a subset of Ω is a *base* if its pointwise stabilizer in G is trivial). Of course, if (G, H) is such a pair then $|H|^2 < |G|$, and it turns out that this condition is almost always sufficient. Clearly, if $H \cap H^x = 1$ for some $x \in G$, for an almost simple primitive group G, then the corresponding action of G on the set of cosets $\Omega = G/H$ is not extremely primitive, so the results in [5] play an essential role in our analysis. In general, to prove that one of the remaining cases (G, H) does not correspond to an extremely primitive group either we apply Lemma 2.2, which gives several sufficient conditions on the point stabilizer H, or we exhibit an explicit element $x \in G$ such that $H \cap H^x$ is not maximal in H. For classical groups of small order, it is convenient to use the computer packages GAP [9] and Magma [3] for direct calculation.

This paper is organized as follows. In Section 2 we fix our notation and we record some preliminary results which will be useful in the proof of Theorem 1.1. The proof itself is given in Sections 3–11, where we partition the analysis according to the 10 subgroup collections listed in Table 2. More precisely, in Section 3 we handle the maximal reducible subgroups of G, which comprise the $\mathcal{C}_1$ collection. Next, in Sections 4 and 5 we consider the subgroups in the $\mathcal{C}_2$ and $\mathcal{C}_3$ collections, while the tensor product subgroups (comprising the $\mathcal{C}_4$ and $\mathcal{C}_7$ families) are quickly dealt with in Section 6. In Section 7 we prove Theorem 1.1 in the case where H is a subfield subgroup, and the subgroups in $\mathcal{C}_6$ and $\mathcal{C}_8$ are handled in Sections 8 and 9, respectively. Finally, we deal with the subgroups in the remaining $\mathcal{C}_9$ and $\mathcal{C}_{10}$ collections in Sections 10 and 11.

2. Preliminaries

2.1. Notation

We start by fixing some of the notation we will use throughout the paper, most of which is standard. Let G be a finite group and let n be a positive integer. Then Z_n and D_n denote the cyclic and dihedral groups of order n, respectively, and we write $[n]$ for an unspecified solvable group of order n. By G^n we denote the direct product of n copies of G, and $\mathrm{Soc}(G)$ is the socle of G (the product of the minimal normal subgroups of G). In addition, we use $Z(G)$ and $F(G)$ to denote the centre and the Fitting subgroup of G, respectively, while $\mathbb{F}_q$ is the field of q elements. For integers a and b, (a, b) denotes the highest common factor of a and b, $\delta_{a,b}$ is the familiar Kronecker delta, and $\mathrm{M}_{a\times b}(k)$ is the set of $a \times b$ matrices over the field k.

As previously remarked, we adopt the standard notation of [14] for classical groups. There are several exceptional isomorphisms between the low-dimensional classical groups:

$$\Omega_3(q) \cong \mathrm{PSp}_2(q) \cong \mathrm{PSU}_2(q) \cong \mathrm{PSL}_2(q), \qquad \mathrm{P}\Omega_4^-(q) \cong \mathrm{PSL}_2(q^2), \qquad \Omega_5(q) \cong \mathrm{PSp}_4(q), \qquad \mathrm{P}\Omega_6^{\varepsilon}(q) \cong \mathrm{PSL}_4^{\varepsilon}(q)$$

(see [14, Proposition 2.9.1]). Consequently, if G_0 is a simple classical group with natural module of dimension n then we will assume $n \geq 3$ if G_0 is unitary, $n \geq 4$ if G_0 is symplectic, and $n \geq 7$ if G_0 is orthogonal. In addition, if q is even then $\Omega_{2m+1}(q) \cong \mathrm{PSp}_{2m}(q)$ for all $m \geq 1$, whence we will assume q is odd if G_0 is an odd dimensional orthogonal group.

Finally, a note on our terminology for automorphisms. Let L be a finite simple group of Lie type. By a theorem of Steinberg [20, Theorem 30], every automorphism of L is a product of the form $idfg$, where i is an *inner* automorphism of L, d a *diagonal* automorphism, and f and g are *field* and *graph* automorphisms of L, respectively. In this paper we adopt the terminology of [10, Definition 2.5.13] for the various types of automorphisms of L.

2.2. Preliminary results

Let G be a primitive permutation group on a finite set Ω with point stabilizer H. Recall that a subset B of Ω is a *base* for G if the pointwise stabilizer of B in G is trivial; we write $b(G)$ for the minimal size of a base for G. Determining $b(G)$ is an interesting problem, with important applications in computational group theory (see [19, Chapter 4], for example). Bases for almost simple classical groups are studied in [4,5], and the examples which admit a base of size two are determined in [5]. Of course, if $b(G) = 2$ then $H \cap H^x = 1$ for some $x \in G$, and thus G is not extremely primitive (note that a maximal subgroup of an almost simple group cannot be of prime order). This trivial observation, combined with the main theorem of [5], plays an essential role in our analysis.

Lemma 2.1. *Let G be an almost simple permutation group, and let $b(G)$ be the minimal size of a base for G. If $b(G) = 2$ then G is not extremely primitive.*

The next lemma provides four conditions on the point stabilizer H, each of which implies that G is not extremely primitive.

Lemma 2.2. *Suppose $|H|$ is composite and one of the following conditions hold:*

(i) $Z(H) \neq 1$.
(ii) *$F(H)$ is not elementary abelian.*
(iii) *$F(H)$ is an elementary abelian group Z_p^e, but $|\Omega| - 1$ is indivisible by p^e.*
(iv) *$F(H)$ is an elementary abelian group Z_p^e, but $H/F(H)$ is not isomorphic to a subgroup of $\mathrm{GL}_e(p)$.*

Then G is not extremely primitive.

Proof. First recall Manning's theorem: if G is extremely primitive then $H = G_\alpha$ is faithful on each of its orbits in $\Omega \setminus \{\alpha\}$ (see [18]). Now, if either $Z(H) \neq 1$ or $F(H)$ is not a p-group for some prime p then H cannot have a faithful primitive permutation representation. Now suppose the Fitting subgroup $F(H)$ is a p-group and let $E \cong Z_p^e$ be an elementary abelian characteristic subgroup of H. Then all primitive faithful permutation representations of H are of affine type of degree p^e, so if $|\Omega| - 1$ is indivisible by p^e, or if H/E is not isomorphic to a subgroup of $\mathrm{GL}_e(p)$, then G is not extremely primitive. Finally, if $F(H) \neq E$ then H cannot have a primitive faithful permutation representation of degree p^e because the point stabilizers in such a representation, considered as subgroups of $\mathrm{GL}_e(p)$, would have nontrivial normal p-subgroups, and hence would not act irreducibly on the vector space $\mathbb{F}_p^e$. □

Lemma 2.3. *Let H_0 be a simple group of Lie type over a finite field of order a power of a prime p, and let H be an extension of H_0 by a subgroup of the group generated by the diagonal and field automorphisms of H_0. Let K be a subgroup of H containing a Sylow p-subgroup of H_0 such that $K \cap H_0$ is properly contained in a maximal parabolic subgroup of H_0. Then K is not maximal in H.*

Proof. Let S be a Sylow p-subgroup of H_0 contained in K, so $S \leqslant K_0$ where $K_0 = K \cap H_0$. Since H_0 is normal in H, it follows that K_0 is normal in K, and so by the Frattini argument, $K = K_0N_K(S)$. Now $H_0N_K(S)$ properly contains $K_0N_K(S) = K$, so if $H_0N_K(S) \neq H$ then K is not maximal in H. Thus we may assume that $H = H_0N_K(S)$.

Let M_0 be a maximal parabolic subgroup of H_0 properly containing K_0. Then M_0 contains a Borel subgroup B of H_0 containing S, and B is a normal subgroup of $N_H(S)$. Moreover the maximal subgroups of H_0 containing B form a set of pairwise non-conjugate maximal parabolic subgroups P_J of H_0, in one-to-one correspondence with maximal proper subsets J of vertices of the corresponding Dynkin diagram of H_0, see [7, Theorems 8.3.2 and 8.3.3]. Since H contains only diagonal and field automorphisms of H_0, $N_H(S)$ normalizes each maximal parabolic subgroup P_J containing B. In particular, M_0 is $N_K(S)$-invariant.

Set $M = M_0N_K(S)$. Then M contains $K_0N_K(S) = K$. Also, since $H = H_0N_K(S)$ it follows that $H = H_0M = H_0K$ and hence $|H : H_0| = |M : M_0| = |K : K_0|$. This implies that $|M : K| = |M_0 : K_0|$ and $|H : M| = |H_0 : M_0|$, and hence K is not maximal in H. □

3. Reducible subgroups

Let G be an almost simple classical group over $\mathbb{F}_q$ with socle G_0 and natural module V of dimension n, where $q = p^f$ for a prime p. Write $G_0 = \Omega(V)/Z$ where Z is the centre of the quasisimple group $\Omega(V)$, and let $\mathrm{I}(V)$ denote the full isometry group of the appropriate $\Omega(V)$-invariant non-degenerate form on V, or $\mathrm{GL}(V)$ if $G_0 = \mathrm{PSL}(V)$. In fact, in the linear case we equip V with the trivial all-zero form, and regard every subspace of V as totally singular.

We begin the proof of Theorem 1.1 by considering the subgroups in Aschbacher's $\mathcal{C}_1$ collection, comprising the stabilizers in G of non-degenerate or totally singular subspaces of V, or pairs of subspaces in the linear case. In addition, if G is an orthogonal group and $p = 2$ then we also consider the stabilizers of 1-dimensional non-singular subspaces of V. The list of cases to be considered is given in [14, Table 4.1.A]. Recall that we may assume $n \geq 2, 3, 4, 7$ in the case of linear, unitary, symplectic, and orthogonal groups, respectively.

Let $H \in \mathcal{C}_1$ be a maximal subgroup of G and let $\Omega = G/H$ be the primitive G-set of right cosets of H in G. The action of G on Ω is permutation isomorphic to the action of $\hat{G}$ on the set of right cosets of a maximal subgroup $M < \hat{G}$, where $\hat{G}$ is the appropriate 'lift' of G containing $\Omega(V)$. Therefore, for the purpose of determining whether or not the action of G on Ω is extremely primitive, we may replace G by $\hat{G}$, and H by M.

Lemma 3.1. *Let G be a symplectic, unitary or orthogonal group. Then G acts transitively on the set of orthogonal decompositions of V as a sum of two non-degenerate subspaces of given dimension (and, in the orthogonal case, of given type).*

Proof. Suppose $V = U_1 \perp W_1 = U_2 \perp W_2$, where U_1 and U_2 are non-degenerate subspaces of the same dimension and type. By Witt's Lemma (see [2, Section 20], for example), there exists $g \in \mathrm{I}(V)$ with $U_1^g = U_2$. Moreover, since $S = \mathrm{I}(U_2) \times \mathrm{I}(U_2^{\perp})$ is the stabilizer of U_2 in the full isometry group $\mathrm{I}(V)$, we have $\mathrm{I}(V) = \Omega(V)S$ and hence there exists $h \in S$ such that $gh \in \Omega(V)$ and $U_1^{gh} = U_2$. □

Proposition 3.2. *Let G be a symplectic, unitary or orthogonal group, and let $H = G_U$ be the G-stabilizer of a non-degenerate k-subspace U of V with $k \leq n/2$. Then G is not extremely primitive.*

Proof. Here $V = U \perp U^{\perp}$ and Lemma 3.1 implies that the permutation domain Ω of G can be identified with the set of non-degenerate k-dimensional subspaces of V. Since H is maximal in G, either $k < n/2$, or $G_0 = \mathrm{P}\Omega_n^-(q)$, $k = n/2$ is even and $\Omega_k^-(q) \times \Omega_k^+(q) \leqslant H$. In any case, we have $\Omega(U) \times \Omega(U^{\perp}) \leqslant H$ (see [14, Lemma 4.1.1(ii)]).

If $Z(\Omega(U)) \neq 1$ or $Z(\Omega(U^{\perp})) \neq 1$ then $Z(H) \neq 1$ and thus G is not extremely primitive by Lemma 2.2(i). Suppose these centres are trivial. If $\Omega(U) \neq 1$ then the socle of H is not the product of isomorphic simple groups, again implying that G is not extremely primitive. The only classical groups with $\Omega(U) = 1$ are the 1-dimensional orthogonal groups, so we have reduced to the case where G is orthogonal and $k = 1$. Further, since U is non-degenerate, we note that q is odd.

Let $U = \langle u \rangle$ and let Q denote the underlying non-degenerate quadratic form on V. Let W be a 2-dimensional anisotropic subspace of V containing $\langle u \rangle$, so $Q(w) \neq 0$ for all non-zero $w \in W$. Then $W \cap U^{\perp} = \langle v \rangle$ for some $v \in V$. Since q is odd, $\langle v \rangle \neq \langle u \rangle$ and we may also choose a third subspace $\langle w \rangle$ of W, different from $\langle u \rangle$ and $\langle v \rangle$. Let $G_{\langle u \rangle, \langle w \rangle}$ and $G_{\langle u \rangle, W}$ denote the subgroups $G_{\langle u \rangle} \cap G_{\langle w \rangle}$ and $G_{\langle u \rangle} \cap G_W$, respectively, so we have

$$G_{\langle u \rangle, \langle w \rangle} \leqslant G_{\langle u \rangle, W} \leqslant G_{\langle u \rangle} = H. \tag{1}$$

Clearly, the inclusion $G_{\langle u \rangle, W} \leqslant G_{\langle u \rangle}$ is proper. We claim that the first inclusion is also proper, proving that G is not extremely primitive. Indeed, $G_{\langle u \rangle, \langle w \rangle}$ acts trivially on W while $G_{\langle u \rangle, W}$ moves every 1-subspace of W different from $\langle u \rangle$ and $\langle v \rangle$, because G_W^W is permutation isomorphic to $D_{2(q+1)}$ on its natural domain of $q + 1$ points. □

Proposition 3.3. *Let G be an orthogonal group with n, q even, and let $H = G_U$ be the G-stabilizer of a non-singular 1-dimensional subspace U of V. Then G is not extremely primitive.*

Proof. We proceed as in the final paragraph of the proof of Proposition 3.2. Let $U = \langle u \rangle$ and let W be a 2-dimensional anisotropic subspace of V containing U. Then $W \cap U^{\perp} = \langle u \rangle$ and $G_W^W \cong D_{2(q+1)}$ acts on an odd number of points, so $G_{\langle u \rangle, W}$ moves every point $\langle w \rangle \neq \langle u \rangle$ in W. Therefore (1) holds and both of the inclusions are proper. The result follows. □

Next we turn to the stabilizers of totally singular subspaces (recall that in the case of linear groups, all subspaces are considered totally singular). Here our analysis relies on the following lemma, which describes precisely when the unipotent radical of such a subgroup is elementary abelian.

Lemma 3.4. *Let $H = G_U$ be the G-stabilizer of a totally singular k-subspace U of V, where $k \leq n/2$. Then the unipotent radical R_H of H is elementary abelian if and only if one of the following holds:*

(i) *G is linear.*
(ii) *G is symplectic, q is even and $k = 1$.*
(iii) *G is orthogonal and $k = 1$.*
(iv) *$k = n/2$.*

Proof. First consider the linear case. We may assume that $U = \langle e_1, \dots, e_k\rangle$, for the first k vectors e_i of a basis of V. With respect to such a basis, the elements of R_H have matrix form

$$X = \begin{pmatrix} I_k & 0 \\ A & I_{n-k} \end{pmatrix}$$

where A is an arbitrary matrix over $\mathbb{F}_q$ of size $(n-k)\times k$, and I_m denotes the m-dimensional identity matrix. It is clear that such matrices commute and have order p, where p is the characteristic of the underlying field $\mathbb{F}_q$.

Now assume G is a symplectic, unitary or orthogonal group. Set $\mathbb{F} = \mathbb{F}_{q^2}$ in the unitary case and $\mathbb{F} = \mathbb{F}_q$ in the other two cases. We may assume that $U = \langle e_1, \dots, e_k\rangle$ and $V/U^{\perp} = \langle f_1 + U^{\perp}, \dots, f_k + U^{\perp}\rangle$, where $e_1, \dots, e_k, f_1, \dots, f_k$ are part of a *standard basis* for V (in the sense of [14, Chapter 2]), so the underlying sesquilinear form β on V takes the following values:

$$\beta(e_i, e_j) = \beta(f_i, f_j) = 0, \qquad \beta(e_i, f_j) = \delta_{i,j}$$

for all $1 \le i, j \le k$, where $\delta_{i,j} = 0$ if $i \neq j$, and 1 if $i = j$. We extend this basis for U to an ordered basis

$$\mathcal{B} = (e_1, \dots, e_k, v_1, \dots, v_{n-2k}, f_k, \dots, f_1)$$

for V so that $U^{\perp} = \langle e_1, \dots, e_k, v_1, \dots, v_{n-2k}\rangle$. In terms of this basis, the elements $X \in R_H$ are of the form

$$X = \begin{pmatrix} I_k & 0 & 0 \\ A & I_{n-2k} & 0 \\ B & C & I_k \end{pmatrix} \tag{2}$$

where A, B, C are matrices over $\mathbb{F}$ of dimensions $(n-2k)\times k$, $k\times k$, $k\times(n-2k)$, respectively. Moreover, we may choose the v_i so that $\mathcal{B}$ is *standard* in the sense of [14, Propositions 2.3.2, 2.4.1, 2.5.3], so the matrix representing the sesquilinear form with respect to $\mathcal{B}$ will have shape

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & J \\ 0 & K & 0 \\ J' & 0 & 0 \end{pmatrix}$$

and the submatrices J, J' and K have the following properties:

(i) $J \in \mathrm{M}_{k\times k}(\mathbb{F})$, where $J_{ij} = 1$ if $i + j = k + 1$, otherwise $J_{ij} = 0$;
(ii) $J' = \delta' J$, where $\delta' = -1$ if G is symplectic, and $\delta' = 1$ in the unitary and orthogonal cases;
(iii) K is the matrix of the form induced on $U^{\perp}/U$ relative to the ordered basis $(v_1 + U, \dots, v_{n-2k} + U)$. This matrix satisfies $K^T = \delta' K$ in the symplectic and orthogonal cases (with δ' as in (ii)), while $K^T = \bar{K} = K$ if G is unitary.

Here X^T denotes the transpose of a matrix X and, for a matrix $X = (X_{ij})$ over $\mathbb{F}_{q^2}$, $\bar{X}$ denotes its image under the Frobenius map $(X_{ij}) \mapsto (X_{ij}^q)$.

The condition that a matrix $X \in \mathrm{M}_{n\times n}(\mathbb{F})$ preserves the form defined by $\mathcal{J}$ is that $\mathcal{J} = X\mathcal{J}X^T$ in the symplectic or orthogonal cases, and $\mathcal{J} = X\mathcal{J}\bar{X}^T$ in the unitary case. For a matrix X as in (2), this is equivalent to requiring that the following two conditions hold:

	Symplectic/Orthogonal case	Unitary case
(I)	$J'A^T = -CK$	$J'\bar{A}^T = -CK$
(II)	$-CKC^T = J'B^T + BJ$	$-CK\bar{C}^T = J'\bar{B}^T + BJ$

Satisfying (I) and (II) is equivalent to being in R_H in the symplectic, unitary, and odd characteristic orthogonal cases. However, if G is orthogonal with n even and $p = 2$ then (I) and (II) are only necessary conditions—in addition, X must also preserve the quadratic form on V defined by

$$Q : (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n/2} x_i x_{n+1-i}. \tag{3}$$

Two elements

$$X_1 = \begin{pmatrix} I_k & 0 & 0 \\ A_1 & I_{n-2k} & 0 \\ B_1 & C_1 & I_k \end{pmatrix}, \qquad X_2 = \begin{pmatrix} I_k & 0 & 0 \\ A_2 & I_{n-2k} & 0 \\ B_2 & C_2 & I_k \end{pmatrix} \tag{4}$$

of R_H commute if and only if $C_2A_1 = C_1A_2$. By using (I) to express C in terms of J', A and K (using the fact that J' and K are both invertible), we deduce that this commutativity criterion is equivalent to the conditions

$$A_2^T K^{-1} A_1 = A_1^T K^{-1} A_2, \qquad \bar{A_2}^T K^{-1} A_1 = \bar{A_1}^T K^{-1} A_2 \tag{5}$$

in the symplectic/orthogonal and unitary cases, respectively.

If $k = n/2$ then (5) is satisfied vacuously, and it is also clear that R_H is elementary abelian. Now assume $k < n/2$. We claim that *any matrix* $A \in \mathrm{M}_{(n-2k)\times k}(\mathbb{F})$ *may occur in the* $(2, 1)$ *block position of an element of* R_H.

To see this, first observe that any given matrix A determines C uniquely by (I), so by (II), the entries b_{ij} of B can be chosen arbitrarily for $i+j < k+1$, and b_{ij} determines $b_{k+1-j,k+1-i}$ uniquely. In the symplectic case, the entries $b_{i,k+1-i}$ cancel out in (II) and so they are arbitrary, whereas in the unitary case, (II) gives q solutions for each $b_{i,k+1-i}$. Similarly, if G is orthogonal and q is odd then (II) determines $b_{i,k+1-i}$ uniquely. Therefore, to establish the claim we may assume G is orthogonal and $p = 2$.

Here the $b_{i,k+1-i}$ cancel out in (II), but we claim that respecting the quadratic form Q defined in (3) determines them uniquely. To see this, suppose G is orthogonal and assume that X_1, X_2 in (4) satisfy $A_1 = A_2$ (and hence $C_1 = C_2$) and the entries with indices $i+j < k+1$ coincide in B_1 and B_2. Then

$$X_1X_2^{-1} = \begin{pmatrix} I_k & 0 & 0 \\ 0 & I_{n-2k} & 0 \\ B_1 - B_2 & 0 & I_k \end{pmatrix}$$

where all entries of $B_1 - B_2$ not on the off-diagonal $(i, k+1-i)$ are equal to 0. Denote the entry of $B_1 - B_2$ in position $(i, k+1-i)$ by b_i. Taking the images of $e_1, \dots, e_k$ under $X_1X_2^{-1}$, (3) implies that

$$0 = Q(e_i) = Q(e_iX_1X_2^{-1}) = b_i \cdot 1,$$

so $b_i = 0$ for all i. Hence, for any A_1 and for any 'upper-half' of B, there is at most one element $X \in R_H$ with these entries. The number of possibilities for A_1 and the upper-half of B is $q^{k(n-2k)+k(k-1)/2}$ and by [14, Proposition 4.1.20], this number is equal to $|R_H|$. Hence for each A_1 and for each upper-half of B, there is exactly one solution. This justifies the claim.

Let $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ be the sequence of columns in A_1 and A_2, respectively, for two matrices $X_1, X_2 \in R_H$ as in (4). By the above claim, if $k \geq 2$ then we may choose

$$x_1^T = (1, 0, 0, \dots, 0), \qquad y_1^T = (0, 0, \dots, 0), \qquad y_2^T = (0, 0, \dots, 0, 1).$$

Then the $(1, 2)$ positions of the products on the two sides of the equations in (5) are zero and non-zero, respectively, so R_H is nonabelian. Finally, suppose $k = 1$. If G is symplectic we set

$$x_1^T = (1, 0, 0, \dots, 0), \qquad y_1^T = (0, 0, \dots, 0, 1),$$

in which case (5) yields the equation $1 = -1$, so $p = 2$ is the only possibility. Similarly, if G is unitary then we may choose $x_1^T = (1, 0, 0, \dots, 0)$ and $y_1^T = (0, 0, \dots, 0, \omega)$ with $\mathbb{F}_{q^2}^* = \langle \omega \rangle$, so $\omega = \omega^q$ from (5), a contradiction. Finally, if G is orthogonal, or if G is symplectic and $p = 2$, then it is straightforward to check that R_H is elementary abelian. $\square$

We also need the following number-theoretical lemma.

Lemma 3.5. *Let q be a prime power and let $n > k \geq 1$ be integers. Then*

$$\frac{\prod_{i=1}^{k}(q^{n+1-i} - 1)}{\prod_{i=1}^{k}(q^{k+1-i} - 1)} \equiv q + 1 \pmod{q^2}.$$

Proof. For fixed k and q, we proceed by induction on n. Let

$$f(n) = \prod_{i=1}^{k}(q^{n+1-i} - 1) \prod_{i=1}^{k}(q^{k+1-i} - 1)^{-1}.$$

The base case is $f(k+1) = (q^{k+1} - 1)/(q - 1)$ which is obviously congruent to $q + 1$ mod q^2. Suppose $f(n) \equiv q + 1 \pmod{q^2}$. Then

$$f(n+1) - f(n) = \left[(q^{n+1} - 1) - (q^{n+1-k} - 1)\right] \frac{\prod_{i=1}^{k-1}(q^{n+1-i} - 1)}{\prod_{i=1}^{k}(q^{k+1-i} - 1)} = q^{n+1-k}\frac{A}{B}$$

for some integers A, B, where q does not divide B. Therefore q^2 divides $f(n+1) - f(n)$ since $n > k$, so $f(n+1) \equiv q + 1 \pmod{q^2}$ as required. $\square$

Proposition 3.6. *Let $H = G_U$ be the G-stabilizer of a totally singular k-subspace U of V, where $k \leq n/2$. Then G is extremely primitive if and only if $n = 2$, $k = 1$ and $G_0 = \mathrm{PSL}_2(q)$, as in line 1 of Table 1.*

Proof. With one exception, the permutation domain Ω of G can be identified with the set of k-dimensional totally singular subspaces of V; the only exception is when $G_0 = \mathrm{P}\Omega_n^+(q)$ and $k = n/2$. In this latter case, the maximality of H implies that $\Omega = U^G$ consists of those subspaces W such that $U \cap W$ has even codimension in both U and W (so Ω contains half of the totally singular k-subspaces of V).

In all cases, the unipotent radical R of H is nontrivial. If R is not elementary abelian then G cannot be extremely primitive by Lemma 2.2(ii). According to Lemma 3.4, R is elementary abelian if and only if one of the following hold:

(i) $G_0 = \mathrm{PSL}_n(q)$, k arbitrary. In this case,

$$|\Omega| = \frac{\prod_{i=1}^{k}(q^{n+1-i}-1)}{\prod_{i=1}^{k}(q^{k+1-i}-1)}$$

and $|R| = q^{k(n-k)}$.

(ii) $G_0 = \mathrm{PSp}_n(q)$, $k = n/2$. Here $|\Omega| = \prod_{i=1}^{k}(q^i+1)$ and $|R| = q^{k(k+1)/2}$.

(iii) $G_0 = \mathrm{PSp}_n(q)$, $p = 2$, $k = 1$. In this case, $|\Omega| = (q^n-1)/(q-1)$ and $|R| = q^{n-1}$.

(iv) $G_0 = \mathrm{P}\Omega_n^{\varepsilon}(q)$, $k = 1$. If n is odd then $|\Omega| = (q^{n-1}-1)/(q-1)$, otherwise $|\Omega| = (q^{n/2}-\varepsilon)(q^{n/2-1}+\varepsilon)/(q-1)$. In all cases $|R| = q^{n-2}$.

(v) $G_0 = \mathrm{P}\Omega_n^+(q)$, $k = n/2$. In this case $|\Omega| = \prod_{i=1}^{k-1}(q^i+1)$ (see the opening paragraph of the proof) and $|R| = q^{k(k-1)/2}$.

(vi) $G_0 = \mathrm{PSU}_n(q)$, $k = n/2$. Here $|\Omega| = \prod_{i=1}^{k}(q^{2i-1}+1)$ and $|R| = q^{k^2}$.

In all six cases, $|\Omega| \equiv q+1 \pmod{q^2}$. This follows from Lemma 3.5 in case (i), and from trivial calculations in the other cases. Hence, by Lemma 2.2(iii), if $|R| > q$ then G is not extremely primitive. Since we assumed that $n \geq 3, 4, 7$ in the unitary, symplectic and orthogonal cases, respectively, the condition $|R| = q$ implies that $G_0 = \mathrm{PSL}_2(q)$ with G acting on $q+1$ points (so H is a Borel subgroup of G). This possibility indeed gives 2-transitive, extremely primitive examples, and we record this case in Table 1, line 1. □

Proposition 3.7. *Suppose $G_0 = \mathrm{PSL}_n(q)$ and H is the G-stabilizer of a pair of subspaces $\{U, W\}$ of V, where either $V = U \oplus W$, or $U \subseteq W$ and $\dim U + \dim W = n$. Then G is not extremely primitive.*

Proof. Here G contains a graph automorphism of G_0, and $H \cap G_0$ is not maximal in G_0 (so H is a *novelty* subgroup of G). Set $\tilde{H} = H \cap \mathrm{PGL}(V)$ and let $W_2 \neq W$ be a subspace of V with $\dim W_2 = \dim W$. In addition, let us assume that either $V = U \oplus W_2$, or $U \subseteq W_2$ in the two cases under consideration, respectively. Then there exists $x \in G_0$ with $U^x = U$ and $W^x = W_2$. For this particular element x we have $H \cap H^x \leqslant \tilde{H}$ because no element of H exchanging U and W can also exchange U and W_2. Moreover, the containment $H \cap H^x < \tilde{H}$ is proper because there are elements of $\tilde{H}$ that stabilize W but do not stabilize W_2. Therefore we have a chain of proper subgroups $H \cap H^x < \tilde{H} < H$, and thus G is not extremely primitive. □

4. Imprimitive subgroups

The subgroups of G in Aschbacher's $\mathcal{C}_2$ collection are the stabilizers of direct sum decompositions

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

of the natural G_0-module V, where $k \geq 2$ and $\dim V_i = m$ for all i. We will write $(V_1, \ldots, V_k)$ to denote such a decomposition of V. In the unitary, symplectic and orthogonal cases we require that either the V_i are non-degenerate and pairwise orthogonal, or $k = 2$ and V_1, V_2 are totally singular. See [14, Table 4.2.A] for a complete list of the subgroups in the $\mathcal{C}_2$ family. In all cases the stabilizer permutes the V_i transitively.

Proposition 4.1. *If $m \geq 2$ and $k \geq 3$ then G is not extremely primitive.*

Proof. If G is not linear then the decomposition $\alpha = (V_1, \ldots, V_k)$ is orthogonal with non-degenerate V_i. In all cases, H contains a normal subgroup $N = \prod_{i=1}^{k} H_i$, where H_i is a classical group on V_i and all the H_i are isomorphic and nontrivial. Consider another decomposition

$$\beta = (W_1, W_2, V_3, \ldots, V_k)$$

with $\langle V_1, V_2 \rangle = \langle W_1, W_2 \rangle$ and β orthogonal in the nonlinear cases. By Witt's Lemma, there exists $x \in G$ that maps α to β, so the stabilizer of β in G is H^x.

Suppose that G is extremely primitive. Then H acts faithfully and primitively on its orbit β^H, and hence its normal subgroup N acts faithfully and transitively on β^H. This means in particular that no nontrivial normal subgroup of N fixes an element of β^H. However since $k \geq 3$, H_3 is a nontrivial normal subgroup of N and H_3 fixes β, which is a contradiction. □

Proposition 4.2. *If $m \geq 2$ and $k = 2$ then G is extremely primitive if and only if $G_0 = \mathrm{PSp}_4(2)'$ and the V_i are non-degenerate, as in lines* 1, 2 *of Table 1 with $q = 9$ and with $(n, \varepsilon) = (4, -)$, respectively.*

Proof. We distinguish several cases according to the nature of the blocks in the decomposition $V = V_1 \oplus V_2$ fixed by H. Again write $\mathbb{F} = \mathbb{F}_{q^2}$ if G is unitary, otherwise $\mathbb{F} = \mathbb{F}_q$.

Case 1: *The blocks are totally singular.* First assume V_1 and V_2 are totally singular subspaces. Since $m \geq 2$, it follows in particular that $|H| > 2$. Now G_{V_2} does not fix V_1. Let $x \in G_{V_2}$ such that $W_1 := V_1^x \neq V_1$. Then H^x is the stabilizer of the decomposition (W_1, V_2), and we have $H \cap H^x \leqslant H_{V_2} \leqslant H$. The second inclusion is proper since H interchanges V_1 and V_2. If the first inclusion is proper then this H-action is imprimitive so G is not extremely primitive. If $H \cap H^x = H_{V_2}$ then the corresponding H-orbit has length $|H : H_{V_2}| = 2$ and the kernel of the H-action is $H_{V_2} \neq 1$ (since $|H| > 2$), so again G is not extremely primitive, since in an extremely primitive group each H-action is faithful.

Case 2: *The blocks are non-degenerate and* $|\mathbb{F}| > 2$. Now suppose G is nonlinear and (V_1, V_2) is an orthogonal decomposition and each V_i is non-degenerate. In addition, let us assume $|\mathbb{F}| > 2$. For a subspace U of V let $\mathrm{Rad}(U) = U \cap U^{\perp}$ denote the radical of U. Write $V_i = \langle e_i, f_i \rangle \perp \bar{V}_i$ with $\{e_i, f_i\}$ a hyperbolic pair, and define

$$W_1 = \langle e_1 + e_2, f_1 \rangle \perp \bar{V}_1, \qquad W_2 = \langle e_2, f_1 - f_2 \rangle \perp \bar{V}_2.$$

It is easy to check that W_1 and W_2 are non-degenerate, the indicated decomposition of each W_i is orthogonal, and $V = W_1 \perp W_2$. By Witt's Lemma, there exists $x \in G$ such that H^x is the stabilizer of the orthogonal decomposition (W_1, W_2) of V.

Suppose $g \in H \cap H^x$ and $V_1 g = V_1$. Then $W_1 g = W_1$ because $\dim(V_1 \cap W_1) = m - 1 > 0$ and $\dim(V_1 \cap W_2) = 0$, so g cannot map W_1 to W_2. Hence $(V_1 \cap W_1)g = V_1 \cap W_1$. We also have $\mathrm{Rad}(V_1 \cap W_1) = \langle f_1 \rangle$, so $\langle f_1 \rangle g = \langle f_1 \rangle$. Summarizing, we have $g \in H_{\langle f_1 \rangle}$, say $f_1 g = c_2 f_1$, and $e_1 g = c_1 e_1 + u_1$ for some $u_1 \in \langle f_1 \rangle \perp \bar{V}_1$. Similarly, since $V_2 g = V_2$, $W_2 g = W_2$ and $\langle e_2 \rangle = \mathrm{Rad}(V_2 \cap W_2)$ we deduce that $g \in H_{\langle e_2 \rangle}$, say $e_2 g = c_3 e_2$, and also $f_2 g = c_4 f_2 + u_2$ for some $u_2 \in \langle e_2 \rangle \perp \bar{V}_2$.

We claim that $c_1 = c_3$ and $c_2 = c_4$. Indeed, since $(e_1 + e_2)g = c_1 e_1 + c_3 e_2 + u_1 \in W_1$ and $u_1 \in \langle f_1 \rangle \perp \bar{V}_1 \subseteq W_1$, it follows that $c_1 e_1 + c_3 e_2$ must lie in W_1 and hence must be a scalar multiple of $e_1 + e_2$. Similarly, $(f_1 - f_2)g = c_2 f_1 - c_4 f_2 - u_2 \in W_2$ and $u_2 \in \langle e_2 \rangle \perp \bar{V}_2 \subseteq W_2$, implying that $c_2 f_1 - c_4 f_2$ is a scalar multiple of $f_1 - f_2$.

A similar argument shows that if $g \in H \cap H^x$ and $V_1 g = V_2$ then $W_1 g = W_2$ because $m - 1 = \dim(V_1 \cap W_1) \neq \dim(V_2 \cap W_1) = 0$ and we have $f_1 g \in \langle e_2 \rangle$ because $\mathrm{Rad}(V_1 \cap W_1)$ must be mapped to $\mathrm{Rad}(V_2 \cap W_2)$. Analogously, $e_2 g \in \langle f_1 \rangle$ and thus

$$H \cap H^x \leqslant H_{\{\langle f_1 \rangle, \langle e_2 \rangle\}} < H.$$

We claim that the first inclusion is proper. If equality holds then $H \cap H^x \cap H_{V_1} = H_{\langle f_1 \rangle, \langle e_2 \rangle}$, which is a contradiction because $|\mathbb{F}| > 2$ and thus $H_{\langle f_1 \rangle, \langle e_2 \rangle}$ contains an element h with the property $e_1 h = c_1 e_1 + u_1$ for some $u_1 \in \langle f_1 \rangle \perp \bar{V}_1$ and $e_2 h = c_3 e_2$, with $c_1 \neq c_3$. The result follows.

Case 3: *The blocks are non-degenerate and* $|\mathbb{F}| = 2$. Here $q = 2$ and G is symplectic or orthogonal. First assume G is symplectic, so m is even. If $m = 2$ then $|\Omega| = 10$ and G is an extremely primitive, 2-transitive group. (Since $\mathrm{PSp}_4(2)' \cong \mathrm{PSL}_2(9)$, in Table 1 this example is recorded in line 1 as $G_0 = \mathrm{PSL}_2(9)$ with H of type P_1, and also it is permutationally isomorphic to the example in line 2 with H of type $O_4^-(2)$.) If $m = 4$ then a GAP [9] computation reveals that $|H \cap H^x| = 64$ for some $x \in G$, so if S is a Sylow 2-subgroup of H containing $H \cap H^x$ then $H \cap H^x \leqslant S \leqslant H$. Moreover, both containments in this subgroup chain are proper, so G is not extremely primitive.

Now assume $m > 4$. Write $V_i = W_i \perp \bar{V}_i$, where each W_i is a 4-dimensional non-degenerate subspace. By the above analysis of the case $m = 4$, there exists $x \in \mathrm{Sp}(W_1 \perp W_2)$ such that $H \cap H^x < (\mathrm{Sp}(\bar{V}_1) \times \mathrm{Sp}(\bar{V}_2)).S < H$ for some Sylow 2-subgroup S of $\mathrm{Sp}(W_1 \perp W_2)$. Therefore G is not extremely primitive.

For the remainder, let us assume G is an orthogonal group. Since $k = 2$, the only possibility is $G_0 = \Omega_n^+(2)$ with $n \geq 8$. There are two possibilities for H, depending on the type of the non-degenerate subspaces V_i in the decomposition $V = V_1 \perp V_2$ stabilized by H. First assume the V_i are both plus type subspaces. If $n = 8$ then an easy calculation with Magma [3] shows that there exists $x \in G$ with $H \cap H^x < L < H$ for some subgroup L of H, with proper containments, so G is not extremely primitive. The general case $n > 8$ quickly follows from the $n = 8$ case, by arguing as above in the symplectic case. The same argument also applies when the V_i are minus type spaces. □

To complete our analysis of the imprimitive subgroups we may assume $m = 1$, so $G_0 = \mathrm{PSL}_n^{\varepsilon}(q)$ or $\mathrm{P}\Omega_n^{\varepsilon}(q)$.

Proposition 4.3. *If $m = 1$ and $G_0 = \mathrm{PSL}_n^{\varepsilon}(q)$ then G is not extremely primitive.*

Proof. First assume $\varepsilon = +$. By [11, Theorem 10.1.3], $b(G) = 2$ unless $n = 2$ and $G = \mathrm{PGL}_2(q).\ell$ for some $\ell > 1$, so by Lemma 2.1 we may assume that we are in this exceptional case. Here $H = N_G(D_{2(q-1)})$. If G is extremely primitive then $F(H)$ is elementary abelian (see Lemma 2.2(ii)), so we may assume that $q - 1 = 2^{\ell} - 1$ is a Mersenne prime. There are precisely $q(q+1)/2$ subgroups in $G_0 = \mathrm{PGL}_2(q)$ isomorphic to $D_{2(q-1)}$, each containing $q - 1$ involutions, while there are exactly $q^2 - 1$ involutions in G_0. Hence each involution in G_0 is contained in exactly $q/2$ distinct dihedral subgroups of order $2(q - 1)$. In particular, there are

$$(q-1)(q/2-1) < \frac{1}{2}q(q+1) - 1$$

dihedral subgroups of G_0 intersecting H_0 in a group generated by an involution, so there is some $x \in G_0$ such that $H_0 \cap H_0^x$ contains no involutions. In this case $H_0 \cap H_0^x = 1$ or Z_{q-1}. However, in the latter case we would have $H_0 = H_0^x$, which is false, so we deduce that $H \cap H^x \cap G_0 = 1$. Therefore $|H \cap H^x| \le |G : G_0| = |H|/2(q-1)$, so $|H : H \cap H^x| \ge 2(q-1)$ and thus G is not extremely primitive.

Now suppose $\varepsilon = -$. If $q+1$ is not prime then $F(H)$ is not elementary abelian, so we may assume q is even and $q+1$ is a Fermat prime. By [5, Proposition 3.1] we have $b(G) = 2$ unless $(n, q) = (3, 4)$, or $q = 2$ and $4 \le n \le 7$. It is easy to check that G is not extremely primitive in each of these remaining cases. For instance, if $q = 2$ then

$$|\Omega| = \frac{|\mathrm{SU}_n(2)|}{3^{n-1}n!}$$

(see [14, Proposition 4.2.9]) and $|F(H)|$ is divisible by 3^{n-2}. However, $|\Omega| - 1$ is not divisible by 3^{n-2} when $4 \le n \le 7$, so G is not extremely primitive by Lemma 2.2(iii). □

Proposition 4.4. *If $m = 1$ and $G_0 = \mathrm{P}\Omega_n^{\varepsilon}(q)$ then G is not extremely primitive.*

Proof. Here $n \ge 7$ and the maximality of H implies that $q = p \ge 3$ and $G \leqslant \mathrm{PGO}_n^{\varepsilon}(p)$, so $H \leqslant 2^{n-1}.S_n$. By [5, Proposition 3.1], we have $b(G) = 2$ unless $q = 3$ and $n \le 8$. If $q = 3$ then

$$|\Omega| = \frac{|\mathrm{SO}_n^{\varepsilon}(3)|}{2^{n-1}n!}$$

and $|F(H)|$ is divisible by 2^{n-2}. It is easy to check that $|\Omega| - 1$ is not divisible by 2^{n-2} when $n = 7$ or 8, so the desired conclusion follows via Lemma 2.2(iii), as before. □

5. Field extension subgroups

In this section we assume the point stabilizer H belongs to Aschbacher's $\mathcal{C}_3$ collection of maximal subgroups of G, so H corresponds to a field extension $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$ for some prime r.

Before we consider the various possibilities for G and H, let us give an explicit description of a natural embedding $\mathrm{GL}_m(q^2) < \mathrm{GL}_{2m}(q)$. We start with an $\mathbb{F}_{q^2}$-basis $\{v_1, v_2, \dots, v_m\}$ for the natural $\mathrm{GL}_m(q^2)$-module W. Let

$$f(x) = x^2 - ax - b \in \mathbb{F}_q[x] \tag{6}$$

be an irreducible polynomial and let $u \in \mathbb{F}_{q^2}$ be a root of f. Note that $b \neq 0$ since f is irreducible. Then $f(u^q) = 0$ so $b = -u^{q+1}$ and $a = u + u^q = T(u)$, where $T : \mathbb{F}_{q^2} \to \mathbb{F}_q$ is the familiar trace map defined by $T : \lambda \mapsto \lambda + \lambda^q$. Now $\{1, u\}$ is an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^2}$ and thus $\{v_1, v_2, \dots, v_m, uv_1, uv_2, \dots, uv_m\}$ is an $\mathbb{F}_q$-basis for the natural $\mathrm{GL}_{2m}(q)$-module V.

Suppose $A = (\alpha_{ij}) \in \mathrm{GL}_m(q^2)$ and $\alpha_{ij} = a_{ij} + ub_{ij}$, where $a_{ij}, b_{ij} \in \mathbb{F}_q$. Then

$$A : v_i \mapsto \sum_{j=1}^{m} (a_{ij}v_j + b_{ij}(uv_j))$$

and

$$A : uv_i \mapsto \sum_{j=1}^{m} (a_{ij}(uv_j) + b_{ij}(u^2v_j)) = \sum_{j=1}^{m} (b_{ij}bv_j + (a_{ij} + ab_{ij})(uv_j))$$

since $u^2 = au + b$. Hence, by introducing the matrices $A_0 = (a_{ij})$ and $A_1 = (b_{ij})$, we see that the action of A on V is given by the matrix

$$A = \left(\begin{array}{c|c} A_0 & A_1 \\ \hline bA_1 & A_0 + aA_1 \end{array}\right) \tag{7}$$

with respect to the specific basis ordering $(v_1, v_2, \dots, v_m, uv_1, uv_2, \dots, uv_m)$.

We now begin the case-by-case analysis of the various possibilities for G and H, as listed in [14, Table 4.3.A]. Our first result provides a reduction to the case $r = 2$ (recall that H corresponds to the field extension $\mathbb{F}_{q^r}/\mathbb{F}_q$ for some prime r).

Proposition 5.1. *If $r \ge 3$ then either $b(G) = 2$, or $G_0 = \mathrm{PSp}_6(q)$ and H is of type $\mathrm{Sp}_2(q^3)$.*

Proof. This follows from [5, Proposition 4.1]. □

In view of Lemma 2.1, if $r \ge 3$ then we may assume $G_0 = \mathrm{PSp}_6(q)$ and H is of type $\mathrm{Sp}_2(q^3)$. This special case is dealt with in the next proposition.

Proposition 5.2. *Suppose $G_0 = \mathrm{PSp}_6(q)$ and $H \in \mathcal{C}_3$ is of type $\mathrm{Sp}_2(q^3)$. Then G is not extremely primitive.*

Proof. If $q \leq 3$ then the result is easily checked using Magma [3], so we will assume $q \geq 4$. Here $H \cap G_0 = H_0.\langle \sigma \rangle$, where $H_0 \cong \mathrm{PSp}_2(q^3)$ and σ is a field automorphism of H_0 of order 3 (see [14, Proposition 4.3.10]). Let $W = V_2(q^3)$ be the natural H_0-module and let $\{e_1, f_1\}$ be a symplectic basis for W with respect to the standard non-degenerate symplectic form β' on W with matrix

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

Then $\beta = T\beta'$ is a non-degenerate symplectic bilinear form on the natural G_0-module $V = V_6(q)$ (see [14, p. 111]), where $T : \lambda \mapsto \lambda + \lambda^q + \lambda^{q^2}$ is the trace map from $\mathbb{F}_{q^3}$ to $\mathbb{F}_q$. Let

$$f(x) = x^3 - ax^2 - bx - c \in \mathbb{F}_q[x] \tag{9}$$

be an irreducible polynomial and let $u \in \mathbb{F}_{q^3}$ be a root of f. Since the coefficients of f lie in the subfield $\mathbb{F}_q$ we have $f(u^q) = f(u^{q^2}) = 0$ and thus

$$a = T(u) = u + u^q + u^{q^2}, \qquad b = -T(u^{1+q}) = -(u^{1+q} + u^{q+q^2} + u^{1+q^2}), \qquad c = u^{1+q+q^2}.$$

Now $\{1, u, u^2\}$ is an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^3}$, whence $\{e_1, f_1, ue_1, uf_1, u^2e_1, u^2f_1\}$ is an $\mathbb{F}_q$-basis for V. In addition, using the above relations, we calculate that

$$T(u^2) = a^2 + 2b, \qquad T(u^3) = a^3 + 3ab + 3c, \qquad T(u^4) = a^4 + 2b^3 + 4a^2b + 4ac,$$

whence the matrix J representing the form β on V is given by the block-matrix

$$J = \begin{pmatrix} 3K & aK & (a^2+2b)K \\ aK & (a^2+2b)K & (a^3+3ab+3c)K \\ (a^2+2b)K & (a^3+3ab+3c)K & (a^4+2b^3+4a^2b+4ac)K \end{pmatrix}$$

with respect to the specific basis ordering $(e_1, f_1, ue_1, uf_1, u^2e_1, u^2f_1)$. Now, if $A = (\alpha_{ij}) \in \mathrm{Sp}_2(q^3)$ and $\alpha_{ij} = a_{ij} + ub_{ij} + u^2c_{ij}$ with $a_{ij}, b_{ij}, c_{ij} \in \mathbb{F}_q$, then it is straightforward to check that A acts on V by

$$A = \begin{pmatrix} A_0 & A_1 & A_2 \\ cA_2 & A_0 + bA_2 & A_1 + aA_2 \\ c(A_1 + aA_2) & bA_1 + (ab+c)A_2 & A_0 + aA_1 + (a^2+b)A_2 \end{pmatrix},$$

where $A_0 = (a_{ij})$, $A_1 = (b_{ij})$ and $A_2 = (c_{ij})$.

Case 1: $p = 2$. Here we may assume $a = 0$ and $c = 1$ in (9), so

$$J = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & K \\ 0 & K & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_0 & A_1 & A_2 \\ A_2 & A_0 + bA_2 & A_1 \\ A_1 & bA_1 + A_2 & A_0 + bA_2 \end{pmatrix}. \tag{10}$$

Let

$$x = x^{-1} = \begin{pmatrix} K & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} \in \mathrm{GL}(V)$$

and note that $x \in G_0$ since $xJx^T = J$. We claim that $H_0 \cap H_0^{x^{-1}}$ is a Sylow 2-subgroup of H_0.

Suppose that $A \in H_0$ has the form given in (10), with $A_0 = (a_{ij})$, $A_1 = (b_{ij})$ and $A_2 = (c_{ij})$ as above. Then

$$x^{-1}Ax = \begin{pmatrix} KA_0K & KA_1 & KA_2 \\ A_2K & A_0 + bA_2 & A_1 \\ A_1K & bA_1 + A_2 & A_0 + bA_2 \end{pmatrix}$$

and this matrix has the form given in (10) if and only if $KA_1 = A_1K = A_1$, $KA_2 = A_2K$, $KA_0K + bKA_2 = A_0 + bA_2$ and $bKA_1 + KA_2 = bA_1 + A_2$. These conditions imply that

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} b_{11} & b_{11} \\ b_{11} & b_{11} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} c_{11} & c_{11} \\ c_{11} & c_{11} \end{pmatrix}.$$

In addition, A also satisfies the condition $AJA^T = J$ since $A \in H_0$, and it is easy to see that this holds if and only if $a_{11}^2 + a_{12}^2 = 1$. Therefore

$$H_0 \cap H_0^{x^{-1}} = \{A(a_{11}, a_{12}, b_{11}, c_{11}) \mid a_{11}^2 + a_{12}^2 = 1\}$$

where

$$A(a_{11},a_{12},b_{11},c_{11})=\left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & b_{11} & b_{11} & c_{11} & c_{11}\\ a_{12} & a_{11} & b_{11} & b_{11} & c_{11} & c_{11}\\ \hline c_{11} & c_{11} & a_{11}+bc_{11} & a_{12}+bc_{11} & b_{11} & b_{11}\\ c_{11} & c_{11} & a_{12}+bc_{11} & a_{11}+bc_{11} & b_{11} & b_{11}\\ \hline b_{11} & b_{11} & bb_{11}+c_{11} & bb_{11}+c_{11} & a_{11}+bc_{11} & a_{12}+bc_{11}\\ b_{11} & b_{11} & bb_{11}+c_{11} & bb_{11}+c_{11} & a_{12}+bc_{11} & a_{11}+bc_{11} \end{array}\right).$$

Here $b_{11}, c_{11} \in \mathbb{F}_q$ can be chosen arbitrarily, while there are exactly q possibilities for the ordered pair of elements (a_{11}, a_{12}) satisfying the condition $a_{11}^2 + a_{12}^2 = 1$. It follows that $|H_0 \cap H_0^{x^{-1}}| = q^3$, whence $H_0 \cap H_0^{x^{-1}}$ is a Sylow 2-subgroup of H_0. This justifies the claim.

It follows that $H_0 \cap H_0^{x^{-1}}$ is properly contained in a Borel subgroup M_0 of H_0, where $|M_0| = q^3(q^3-1)$. Therefore Lemma 2.3 implies that $H \cap H^{x^{-1}}$ is not maximal in H, so G is not extremely primitive.

Case 2: $p = 3$. Now suppose q is odd. Here we may take $(a, b) = (0, 1)$ and $c \neq 1$ in (9). First we consider the special case $p = 3$, so

$$J=\begin{pmatrix} 0 & 0 & 2K\\ 0 & 2K & 0\\ 2K & 0 & 2K\end{pmatrix} \quad\text{and}\quad A=\begin{pmatrix} A_0 & A_1 & A_2\\ cA_2 & A_0+A_2 & A_1\\ cA_1 & A_1+cA_2 & A_0+A_2\end{pmatrix}. \tag{11}$$

Define

$$x=\begin{pmatrix} I_2 & 0 & 0\\ 0 & I_2 & 0\\ B & 0 & I_2\end{pmatrix}, \quad \text{where } B=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}$$

and note that $xJx^T = J$, so $x \in G_0$. Suppose $A \in H_0$ is of the form given in (11), with $A_0 = (a_{ij})$, $A_1 = (b_{ij})$ and $A_2 = (c_{ij})$. If $x^{-1}Ax$ has blocks as in (11) then an easy calculation reveals that

$$A_0=\begin{pmatrix} a_{11} & a_{12}\\ 0 & a_{11}\end{pmatrix}, \qquad A_1=\begin{pmatrix} 0 & b_{12}\\ 0 & 0\end{pmatrix}, \qquad A_2=\begin{pmatrix} 0 & c_{12}\\ 0 & 0\end{pmatrix}.$$

Furthermore, we find that $x^{-1}Ax$ fixes the underlying symplectic form β on V if and only if $a_{11}^2 = 1$, whence

$$H_0 \cap H_0^{x^{-1}} = \{A(a_{11}, a_{12}, b_{12}, c_{12}) \mid a_{11}^2 = 1 \text{ and } a_{12}, b_{12}, c_{12} \in \mathbb{F}_q\}$$

(modulo scalars) where

$$A(a_{11},a_{12},b_{12},c_{12})=\left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & 0 & b_{12} & 0 & c_{12}\\ 0 & a_{11} & 0 & 0 & 0 & 0\\ \hline 0 & cc_{12} & a_{11} & a_{12}+c_{12} & 0 & b_{12}\\ 0 & 0 & 0 & a_{11} & 0 & 0\\ \hline 0 & cb_{12} & 0 & b_{12}+cc_{12} & a_{11} & a_{12}+c_{12}\\ 0 & 0 & 0 & 0 & 0 & a_{11} \end{array}\right).$$

By factoring out the centre of order 2 we deduce that $|H_0 \cap H_0^{x^{-1}}| = q^3$ and thus $H_0 \cap H_0^{x^{-1}}$ is a Sylow 3-subgroup of H_0. The previous argument now applies and we deduce that there are no extremely primitive examples.

Case 3: $p \geq 5$. Here

$$J=\begin{pmatrix} 3K & 0 & 2K\\ 0 & 2K & 3cK\\ 2K & 3cK & 2K\end{pmatrix} \quad\text{and}\quad A=\begin{pmatrix} A_0 & A_1 & A_2\\ cA_2 & A_0+A_2 & A_1\\ cA_1 & A_1+cA_2 & A_0+A_2\end{pmatrix}. \tag{12}$$

Fix $\alpha, \beta \in \mathbb{F}_q^*$ such that $3c\alpha - 2\beta = 0$ and define

$$x=\begin{pmatrix} I_2 & 0 & 0\\ 0 & B & 0\\ 0 & C & I_2\end{pmatrix}, \quad \text{where } B=\begin{pmatrix} 1 & \alpha\\ 0 & 1\end{pmatrix} \text{ and } C=\begin{pmatrix} 0 & \beta\\ 0 & 0\end{pmatrix}.$$

One can check that $xJx^T = J$, so $x \in G_0$. Suppose $A \in H_0$ is of the form given in (12), with $A_0 = (a_{ij})$, $A_1 = (b_{ij})$ and $A_2 = (c_{ij})$. We calculate that $x^{-1}Ax$ has blocks as in (12) if and only if all of the following conditions hold:

(a) $a_{21} = b_{21} = b_{22} = c_{21} = c_{22} = 0$
(b) $\alpha b_{11} + \beta c_{11} = 0$

(c) $\alpha(a_{11} - a_{22} + c_{11}) + \beta b_{11} = 0$
(d) $\beta(a_{11} - a_{22}) + \alpha c c_{11} = 0$.

Furthermore, we see that $x^{-1}Ax$ preserves the form β if and only if all the following additional conditions hold:

(i) $a_{22}(3a_{11} + 2c_{11}) = 3$
(ii) $a_{22}(2b_{11} + 3cc_{11}) = 0$
(iii) $a_{22}(2a_{11} + 3cb_{11} + 2c_{11}) = 2$
(iv) $a_{22}(3ca_{11} + 2b_{11} + 5cc_{11}) = 3c$
(v) $a_{22}(2a_{11} + 5cb_{11} + (3c^2 + 2)c_{11}) = 2$.

Note that conditions (iv) and (v) can be deduced from (i)–(iii). Also note that none of the conditions (a)–(d) and (i)–(v) involve the entries a_{12}, b_{12} or c_{12}.

Recall that $\beta = (3c/2)\alpha$, so from (d) above we deduce that $c_{11} = 3(a_{22} - a_{11})/2$ and thus (b) yields $b_{11} = -9(a_{22} - a_{11})/4$. Since (i) holds, it follows that

$$a_{22}(3a_{11} + 3(a_{22} - a_{11})) = 3a_{22}^2 = 3$$

and thus $a_{22} = \pm 1$. Subsequently, (ii) implies that $2b_{11} + 3cc_{11} = 0$, so

$$0 = -\frac{9}{2}(a_{22} - a_{11}) + \frac{9}{2}c(a_{22} - a_{11}) = \frac{9}{2}(c - 1)(a_{22} - a_{11}).$$

Therefore $a_{11} = a_{22}$ since $c \neq 1$, so $b_{11} = c_{11} = 0$.

Consequently, we deduce that

$$H_0 \cap H_0^{x^{-1}} = \{A(a_{11}, a_{12}, b_{12}, c_{12}) \mid a_{11}^2 = 1 \text{ and } a_{12}, b_{12}, c_{12} \in \mathbb{F}_q\}$$

(modulo scalars) where

$$A(a_{11}, a_{12}, b_{12}, c_{12}) = \left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & 0 & b_{12} & 0 & c_{12} \\ 0 & a_{11} & 0 & 0 & 0 & 0 \\ \hline 0 & cc_{12} & a_{11} & a_{12} + c_{12} & 0 & b_{12} \\ 0 & 0 & 0 & a_{11} & 0 & 0 \\ \hline 0 & cb_{12} & 0 & b_{12} + cc_{12} & a_{11} & a_{12} + c_{12} \\ 0 & 0 & 0 & 0 & 0 & a_{11} \end{array}\right).$$

In particular, $H_0 \cap H_0^{x^{-1}}$ is a Sylow p-subgroup of H_0, and so Lemma 2.3 implies that there are no extremely primitive examples. □

Proposition 5.3. *Suppose $G_0 = \mathrm{PSL}_n(q)$ and $H \in \mathcal{C}_3$ is of type $\mathrm{GL}_{n/2}(q^2)$. Then G is extremely primitive if and only if either $G = \mathrm{PSL}_2(4).2$ (which is permutationally isomorphic to the group $\mathrm{PGL}_2(5)$ acting on cosets of $H = P_1$ as in line 1 of Table 1), or $G = \mathrm{PSL}_2(q)$ and $q + 1$ is a Fermat prime, as in line 3 of Table 1.*

Proof. By [14, Proposition 4.3.6], H has a cyclic normal subgroup of order

$$\ell = \frac{(q + 1)(q - 1, n/2)}{(q - 1, n)} > 1.$$

Therefore, if G is extremely primitive then H must have a faithful primitive representation of affine type, so ℓ is prime and $H \leqslant \mathrm{AGL}_1(\ell)$ by Lemma 2.2. This implies that $n = 2$ and either q is odd and $\ell = (q+1)/2$ is prime, or q is even and $\ell = q+1$ is a Fermat prime. In both cases $q > 3$ because G_0 is simple. Set $H_0 = H \cap G_0$.

First assume q is odd, so $H_0 \cong D_{q+1}$. We proceed as in the proof of Proposition 4.3 (the case $\varepsilon = +$). There are precisely $q(q - 1)/2$ subgroups in G_0 isomorphic to D_{q+1}, each containing $(q + 1)/2$ involutions, while there are exactly $q(q + 1)/2$ involutions in G_0. Hence each involution in G_0 is contained in exactly $(q - 1)/2$ distinct dihedral subgroups of order $q + 1$. In particular, there are

$$\frac{1}{2}(q + 1)((q - 1)/2 - 1) < \frac{1}{2}q(q - 1) - 1$$

dihedral subgroups of G_0 intersecting H_0 in a group generated by an involution, so there is some $x \in G_0$ such that $H_0 \cap H_0^x$ contains no involutions. In this case $H_0 \cap H_0^x = 1$ or $Z_{(q+1)/2}$. However, in the latter case we would have $H_0 = H_0^x$, which is false, so we deduce that $H \cap H^x \cap G_0 = 1$. Therefore $|H \cap H^x| \leq |G : G_0| = |H|/(q+1)$, so $q + 1 \leq |H : H \cap H^x|$ and thus G is not extremely primitive.

Now assume q is even and $q + 1$ is a Fermat prime, so $H_0 \cong D_{2(q+1)}$. Here there are $q(q - 1)/2$ subgroups of G_0 isomorphic to $D_{2(q+1)}$, each containing $q + 1$ involutions. Since there are exactly $q^2 - 1$ involutions in G_0, it follows that each one is contained in exactly $q/2$ dihedral subgroups of order $2(q + 1)$. In particular, there are

$$(q + 1)(q/2 - 1) = \frac{1}{2}q(q - 1) - 1$$

dihedral subgroups of G_0 intersecting H_0 in a group generated by an involution. Consequently, every $D_{2(q+1)}$ subgroup of G_0 different from H_0 intersects H_0 in a group of size 2, whence $|H : H \cap H^x| = q + 1$ for all $x \in G_0 \setminus H$, and thus G_0 is extremely primitive. This case is recorded in line 3 of Table 1.

If $q = 4$ then $G = \mathrm{PSL}_2(4).2$ gives an additional extremely primitive example. Since here $G \cong \mathrm{PGL}_2(5)$ and H is isomorphic to a parabolic subgroup of $\mathrm{PGL}_2(5)$, this example occurs in line 1 of Table 1. Now suppose $q = 2^{2^r} > 4$ and $G \neq G_0$. Then $G = G_0.2^s$ for some s with $1 \leq s \leq r$, and $H = Z_{q+1}.Z_{2^{s+1}}$ is a Frobenius group. If $x \in G_0 \setminus H$ then $|H \cap H^x| \leq 2^{s+1}$; moreover, if this inequality is strict then $|H : H \cap H^x| > q + 1$ and G is not extremely primitive. Suppose $Z := H \cap H^x \cong Z_{2^{s+1}}$; let z be a generator of Z and let $y = z^{2^s} \in G_0$ be the involution in Z. Then $C_{G_0}(y)$ is the unique Sylow 2-subgroup S of G_0 containing y, and we have $C_G(y) = SZ$ and $|C_G(y)| = 2^s q$. Clearly $Z \leqslant C_G(z) \leqslant C_G(y)$. Moreover, S can be identified with the additive group of $\mathbb{F}_q$, so z acts as a field automorphism of order 2^s on S and thus $|C_G(z)| = 2^s q^{2^{-s}}$. Hence $|C_G(y) : C_G(z)| = 2^{2^r - 2^{r-s}} > 2^s$ and there exists $w \in C_G(y) \setminus C_G(z)$ such that z^w is different from any of the 2^s elements of Z of order 2^{s+1}. Set $W = H \cap H^{xw}$. We claim that W is not maximal in H. Since $y \in W$, it follows that W is contained in the unique cyclic subgroup of H containing y, that is, $W \leqslant Z$. However, $W \neq Z$ because $z \notin W$. This justifies the claim and we conclude that G is not extremely primitive. □

Proposition 5.4. *Suppose $G_0 = \mathrm{PSp}_4(q)'$ and $H \in \mathcal{C}_3$ is of type $\mathrm{Sp}_2(q^2)$. Then G is extremely primitive if and only if $q = 2$ and $G = G_0$ or $G \cong S_6$. The actions of these groups are permutationally isomorphic to their actions on the cosets of subgroups of type $\mathrm{O}_4^-(2)$ as in line 2 of Table 1.*

Proof. We proceed as in the proof of Proposition 5.2. For now let us assume $G = G_0$. According to [14, Proposition 4.3.10] we have $H = H_0.\langle\sigma\rangle$, where $H_0 \cong \mathrm{PSp}_2(q^2)$ and σ is an involutory field automorphism of H_0 if $q > 2$, and $\sigma = 1$ if $q = 2$. Let $W = V_2(q^2)$ denote the natural $\mathrm{Sp}_2(q^2)$-module and let $\{e_1, f_1\}$ be a symplectic basis for W with respect to the standard non-degenerate symplectic form β' on W with matrix K as in (8). One can check that $\beta = T\beta'$ is a non-degenerate symplectic bilinear form on the natural G_0-module $V = V_4(q)$ (see [14, p. 111]), where $T : \lambda \mapsto \lambda + \lambda^q$ is the trace map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$.

Recall that $u \in \mathbb{F}_{q^2}$ is a root of an irreducible polynomial $f(x) = x^2 - ax - b \in \mathbb{F}_q[x]$ (see (6)). The other root of $f(x)$ is u^q, so $u^q + u = a$. Also recall that each $A \in H_0$ acts on V as a matrix of the form given in (7), with respect to the ordered $\mathbb{F}_q$-basis (e_1, f_1, ue_1, uf_1) for V. Let J be the matrix of the symplectic form β on V, written with respect to the specific basis ordering (e_1, f_1, ue_1, uf_1). Since $\beta = T\beta'$ and $T(u^2) = T(au + b) = a^2 + 2b$ we deduce that

$$J = \begin{pmatrix} 2K & aK \\ aK & (a^2+2b)K \end{pmatrix}, \quad \text{where } K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly, since $u^q = a - u$, for $q > 2$ we have

$$\sigma : e_1 \mapsto e_1,\ f_1 \mapsto f_1,\ ue_1 \mapsto ae_1 - ue_1,\ uf_1 \mapsto af_1 - uf_1$$

and thus

$$\sigma = \begin{pmatrix} I_2 & 0 \\ aI_2 & -I_2 \end{pmatrix}.$$

In particular, if $q > 2$ then H is generated by σ and all invertible matrices A of the form (7) which satisfy the additional relation $AJA^T = J$.

Case 1: $p = 2$. Here we may take $a = 1$ in (6) (so that $T(u) = 1$), whence

$$J = \begin{pmatrix} 0 & K \\ K & K \end{pmatrix}$$

and every $A \in H_0$ is of the form

$$A = \left(\begin{array}{c|c} A_0 & A_1 \\ \hline bA_1 & A_0 + A_1 \end{array}\right) \tag{13}$$

where $A_0 = (a_{ij})$ and $A_1 = (b_{ij})$ are 2×2 matrices. Set

$$x = \begin{pmatrix} I_2 & 0 \\ K & I_2 \end{pmatrix} \tag{14}$$

and note that $x = x^{-1}$ and $xJx^T = J$, so $x \in G_0$. It is straightforward to check that $x^{-1}Ax$ is a matrix of the form (13) if and only if the following conditions hold:

(a) $b_{11} = b_{22}, b_{12} = b_{21}$
(b) $a_{11} + a_{22} = a_{12} + a_{21} = b_{11} + b_{12}$.

In addition, we calculate that $x^{-1}Ax$ fixes β if and only if the following conditions also hold:

(i) $b_{11}(a_{11}+a_{22})+b_{12}(a_{12}+a_{21})+b_{11}^2+b_{12}^2=0$
(ii) $a_{12}a_{21}+a_{11}a_{22}+b(b_{11}^2+b_{12}^2)=1$.

Note that condition (i) follows immediately from (b) above, while (b) implies that (ii) is equivalent to the condition

$$(a_{11}+a_{12})^2+z(a_{11}+a_{12})+bz^2=1, \quad \text{where } z=b_{11}+b_{12}. \tag{15}$$

Summarizing, $A\in H_0\cap H_0^{x^{-1}}$ if and only if $A=A(a_{11},a_{12},b_{11},b_{12})$, where

$$A(a_{11},a_{12},b_{11},b_{12})=\left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ b_{11}+b_{12}+a_{12} & b_{11}+b_{12}+a_{11} & b_{12} & b_{11} \\ \hline bb_{11} & bb_{12} & a_{11}+b_{11} & a_{12}+b_{12} \\ bb_{12} & bb_{11} & a_{12}+b_{11} & a_{11}+b_{12} \end{array}\right) \tag{16}$$

and the field elements $a_{11},a_{12},b_{11},b_{12}$ satisfy (15).

Now, if $b_{11}=b_{12}$ then $z=b_{11}+b_{12}=0$ and (15) is equivalent to the condition $a_{11}+a_{12}=1$. Hence the q^2 elements $\{A(c,c+1,d,d)\mid c,d\in\mathbb{F}_q\}$ are in $H_0\cap H_0^{x^{-1}}$ and they form a subgroup since

$$A(c,c+1,d,d)\cdot A(c',c'+1,d',d')=A(c+c'+1,c+c',d+d',d+d').$$

Therefore $H_0\cap H_0^{x^{-1}}$ is contained in a Borel subgroup M_0 of H_0 (in fact, M_0 is the stabilizer of $\langle e_1+f_1\rangle$) and $|M_0|=q^2(q^2-1)$. We have $|H_0\cap H_0^{x^{-1}}|\leq 2q^3$ because for a fixed $z\in\mathbb{F}_q$ there are q pairs (b_{11},b_{12}) satisfying $z=b_{11}+b_{12}$ and at most 2 values for $a_{11}+a_{12}$ that satisfy (15), and for each of these values there are q compatible pairs (a_{11},a_{12}). If $q\geq 4$ then $2q^3<q^2(q^2-1)$, so $H_0\cap H_0^{x^{-1}}$ is a proper subgroup of M_0. Also, $H_0\cap H_0^{x^{-1}}$ contains a Sylow 2-subgroup of H_0 and thus Lemma 2.3 implies that $H\cap H^{x^{-1}}$ is not maximal in H. If $q=2$ then $H_0\cap H_0^{x^{-1}}=M_0$ and G_0 is indeed extremely primitive. This action is permutationally isomorphic to the G_0-action on the cosets of a subgroup of type $O_4^-(2)$ as in line 2 of Table 1.

Next assume $p=2$ and $G\neq G_0$. If $q=2$ then we get another extremely primitive example when $G\cong S_6$, and again this case appears in line 2 of Table 1. Suppose $q\geq 4$. If G contains graph–field automorphisms then H is not maximal in G (see [1, Section 14]), so we may assume otherwise. In particular, H is an extension of H_0 by field automorphisms and thus Lemma 2.3 implies that $H\cap H^{x^{-1}}$ is not maximal in H, where $x\in G_0$ is the element defined in (14) above. We conclude that G is not extremely primitive.

Case 2: $p>2$. In this case we may choose $a=0$ and $b=\omega$ in (6), where $\mathbb{F}_q^*=\langle\omega\rangle$, so

$$J=\begin{pmatrix} 2K & 0 \\ 0 & 2\omega K\end{pmatrix}.$$

As before, first assume $G=G_0$. Set

$$x=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^{-1} & 0 \\ 1 & -\omega & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad x^{-1}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^{-1} & \omega^{-1} \\ 0 & -\omega & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

and note that $x\in G_0$ since $xJx^T=J$. Let

$$A=\left(\begin{array}{c|c} A_0 & A_1 \\ \hline \omega A_1 & A_0 \end{array}\right)\in H_0,$$

where $A_0=(a_{ij})$ and $A_1=(b_{ij})$. An easy calculation reveals that $x^{-1}Ax$ is a matrix of the form (7) if and only if $a_{11}=a_{22}$, $b_{11}=-b_{22}$ and $a_{21}=b_{21}=0$. In addition, A preserves β if and only if the entries a_{11} and b_{11} also satisfy the condition

$$a_{11}^2-\omega b_{11}^2=1. \tag{18}$$

Summarizing, we have $A\in H_0\cap H_0^{x^{-1}}$ if and only if

$$A=A(a_{11},a_{12},b_{11},b_{12})=\left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & a_{11} & 0 & -b_{11} \\ \hline \omega b_{11} & \omega b_{12} & a_{11} & a_{12} \\ 0 & -\omega b_{11} & 0 & a_{11} \end{array}\right) \tag{19}$$

and the field elements a_{11},b_{11} satisfy (18).

For each pair (a_{11}^*,b_{11}^*) satisfying (18), there are exactly q^2 elements in $H_0\cap H_0^{x^{-1}}$ of the form $A=A(a_{11}^*,a_{12},b_{11}^*,b_{12})$. It follows that $|H\cap H^{x^{-1}}|$ is divisible by q^2, so $H_0\cap H_0^{x^{-1}}$ contains a Sylow p-subgroup of H_0 and it is therefore contained in a Borel subgroup M_0 of H_0 (in fact, M_0 is the stabilizer of $\langle f_1\rangle$). Moreover, there are exactly $q+1$ possibilities for the ordered pair of elements (a_{11},b_{11}) satisfying (18), so by factoring out the centre of order 2 we deduce that $|H_0\cap H_0^{x^{-1}}|=\frac{1}{2}(q+1)q^2$. Since $|M_0|=\frac{1}{2}q^2(q^2-1)$ we deduce that $H_0\cap H_0^{x^{-1}}$ is a proper subgroup of M_0, whence Lemma 2.3 implies that $H\cap H^{x^{-1}}$ is not maximal in H. A further application of Lemma 2.3 gives the same conclusion when $G\neq G_0$. $\square$

Proposition 5.5. *Suppose $G_0 = \mathrm{PSp}_n(q)$ and $H \in \mathcal{C}_3$ is of type $\mathrm{Sp}_{n/2}(q^2)$, where $n > 4$. Then G is not extremely primitive.*

Proof. Here $n = 4m$ with $m \geq 2$ and $H \cap G_0 = H_0.\langle\sigma\rangle$, where $H_0 \cong \mathrm{PSp}_{2m}(q^2)$ and σ is an involutory field automorphism of H_0. Let $W = V_{2m}(q^2)$ denote the natural $\mathrm{Sp}_{2m}(q^2)$-module and let $\{e_i, f_i \mid 1 \leq i \leq m\}$ be a symplectic basis for W with respect to a standard non-degenerate symplectic form β' on W. The embedding of H_0 in G_0 is described in (7), and we note that $\beta = T\beta'$ is a non-degenerate symplectic form on the natural G_0-module $V = V_n(q)$ (see [14, p. 111]). Observe that the decomposition

$$V = \bigoplus_{i=1}^{m} \langle e_i, f_i, ue_i, uf_i\rangle$$

is orthogonal with respect to *both* β and β', where $u \in \mathbb{F}_{q^2}$ is a root of the irreducible polynomial defined in (6).

Write $V = V_1 \perp V_2$ where $V_1 = \langle e_1, f_1, ue_1, uf_1\rangle$ and $V_2 = \langle e_i, f_i, ue_i, uf_i \mid 2 \leq i \leq m\rangle$. The stabilizer of this decomposition in G_0 is a central product $G_1 \circ G_2$ with $G_1 \cong \mathrm{Sp}_4(q)$ and $G_2 \cong \mathrm{Sp}_{4m-4}(q)$, while the corresponding stabilizer in H_0 is $H_1 \circ H_2$ with $H_1 \cong \mathrm{Sp}_2(q^2)$ and $H_2 \cong \mathrm{Sp}_{2m-2}(q^2)$ (see [14, Proposition 4.1.3]).

Set $z = (x, 1) \in G_1 \times G_2$, where $x \in G_1$ is the element defined in (14) and (17), for q even and odd, respectively. For $A \in H_0$ of the form (7), we write A_i, $i = 0, 1$, in the block form

$$A_i = \begin{pmatrix} (A_i)_{11} & (A_i)_{12} \\ (A_i)_{21} & (A_i)_{22} \end{pmatrix},$$

where $(A_i)_{11}$ has size 2×2. It is straightforward to see that $z^{-1}Az$ is a matrix of the form (7) and fixes β if and only if

$$(A_0)_{21} = (A_1)_{21} = 0, \qquad (A_0)_{12} = (A_1)_{12} = 0$$

and the 2×2 matrices $(A_0)_{11}$ and $(A_1)_{11}$ satisfy the conditions described in (15), (16) and (18), (19) in the cases of even and odd q, respectively. Hence, as we calculated in the proof of Proposition 5.4,

$$S \times H_2 \leqslant H_0 \cap H_0^{z^{-1}} \leqslant M_0 \circ H_2, \tag{20}$$

where M_0 is a Borel subgroup of H_1 and S is the unipotent radical of M_0. Thus $H_0 \cap H_0^{z^{-1}} = M_1 \circ H_2$ where $S \leqslant M_1 \leqslant M_0$ and we note that S is characteristic in $H_0 \cap H_0^{z^{-1}}$.

The group $H \cap H^{z^{-1}}$ normalizes $H_0 \cap H_0^{z^{-1}}$, so it must normalize $H_2 \leqslant H_0 \cap H_0^{z^{-1}}$ and S. Consequently, $H \cap H^{z^{-1}}$ must fix the subspace V_2 and its orthogonal complement V_1. Hence

$$H \cap H^{z^{-1}} \leqslant H_{V_1,V_2} < H, \tag{21}$$

where H_{V_1,V_2} is the H-stabilizer of the decomposition $V = V_1 \perp V_2$. Since $H \cap H^{z^{-1}}$ normalizes S, it follows that $H \cap H^{z^{-1}}$ induces on V_1 a subgroup of a parabolic subgroup and in particular $H \cap H^{z^{-1}}$ does not contain H_1. So $H \cap H^{z^{-1}}$ is a proper subgroup of H_{V_1,V_2} and G is not extremely primitive. $\square$

Proposition 5.6. *Suppose $G_0 = \mathrm{P}\Omega_n^{\varepsilon}(q)$ and $H \in \mathcal{C}_3$ is of type $\mathrm{O}_{n/2}^{\varepsilon'}(q^2)$. Then G is not extremely primitive.*

Proof. We may assume $n \geq 8$. By [12], if $(n, \varepsilon) = (8, +)$ then the action of G on G/H is permutation isomorphic to the action of G on G/M, where M is an imprimitive $\mathcal{C}_2$-subgroup of type $O_4^-(q) \times O_4^-(q)$. By Proposition 4.2, G is not extremely primitive so for the remainder we may assume $(n, \varepsilon) \neq (8, +)$. (In fact, the analysis of the case $(n, \varepsilon) = (8, +)$ with $q \leq 3$ is essential to our argument in the general case $n > 8$, so we will deal with these cases directly. Note that we may always assume G does not contain any triality automorphisms (see [12]).)

The possibilities for G and H are given in [14, Table 4.3.A]. We note that if $n \equiv 0 \pmod 4$ then $\varepsilon = \varepsilon'$, and if $n \equiv 2 \pmod 4$ then q is odd and H is of type $O_{n/2}(q^2)$. More precisely, we have $H \cap G_0 = H_0.[c]$ where $H_0 = \mathrm{P}\Omega_{n/2}^{\varepsilon'}(q^2)$ is simple (since $(n, \varepsilon) \neq (8, +)$), and where $c = 4$ if $\varepsilon = \varepsilon' = +$, otherwise $c = 2$ (see [14, Propositions 4.3.14, 4.3.16, 4.3.20]). We handle all possibilities simultaneously.

Let $W = V_{n/2}(q^2)$ denote the natural $O_{n/2}^{\varepsilon'}(q^2)$-module and let Q' and β' respectively denote the corresponding non-degenerate quadratic form and symmetric bilinear form on W. Fix a basis $\{e_i, f_i \mid 1 \leq i \leq m\} \cup B$ for W so that the $\{e_i, f_i\}$ are pairwise orthogonal hyperbolic pairs. Here B is empty if $\varepsilon' = +$, while $B = \{h_1, h_2\}$ spans a 2-dimensional anisotropic subspace orthogonal to all e_i, f_i when $\varepsilon' = -$. Also, if $n/2$ is odd then $B = \{h\}$ is non-singular and orthogonal to all e_i, f_i. Let $u \in \mathbb{F}_{q^2}$ be a root of the irreducible polynomial defined in (6) and note that we may choose $a = 1$ when q is even, and $(a, b) = (0, \omega)$ when q is odd, where $\mathbb{F}_q^* = \langle\omega\rangle$. Now

$$\{e_i, f_i, ue_i, uf_i \mid 1 \leq i \leq m\} \cup \{h, uh \mid h \in B\}$$

is an $\mathbb{F}_q$-basis for the natural G_0-module $V = V_n(q)$, and the action of elements in $H \cap G_0$ on V is described in (7). In addition, $Q = TQ'$ is a non-degenerate quadratic form on V, with associated symmetric bilinear form $\beta = T\beta'$ (see [14, p. 111]).

Consider the direct sum decomposition $V = V_1 \oplus V_2$ with

$$V_1 = \langle e_1, f_1, ue_1, uf_1 \rangle, \qquad V_2 = \langle e_i, f_i, ue_i, uf_i, h, uh \mid 2 \le i \le m, h \in B \rangle$$

and note that this decomposition is orthogonal with respect to *both* β and β'. Let G_0^* be the group induced on V_1 by the G_0-stabilizer of V_1. Similarly, let H_0^* be the corresponding group induced by the H_0-stabilizer of V_1. We claim that *G_0^* is of type $O_4^+(q)$ and H_0^* is of type $O_2^+(q^2)$.*

Since $Q'(e_1) = Q'(f_1) = 0$, the non-degenerate 2-dimensional orthogonal space $\langle e_1, f_1 \rangle$ contains non-zero singular vectors for Q', so H_0^* is of type $O_2^+(q^2)$. Now consider G_0^*. Proceeding as in the proof of Proposition 5.4, we find that the matrix J representing the restriction of β to V_1 is

$$J = \begin{pmatrix} 0 & K \\ K & K \end{pmatrix} \quad \text{for } q \text{ even}, \qquad J = \begin{pmatrix} 2K & 0 \\ 0 & 2\omega K \end{pmatrix} \quad \text{for } q \text{ odd}, \tag{22}$$

with respect to the basis ordering (e_1, f_1, ue_1, uf_1), where

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If q is even then $\{e_1, uf_1\}$ and $\{e_1 + ue_1, f_1\}$ are orthogonal hyperbolic pairs. For instance, we have

$$Q(e_1 + ue_1) = TQ'((1+u)e_1) = T((1+u)^2 Q'(e_1)) = 0.$$

Similarly, if q is odd then $\{e_1, \frac{1}{2}f_1\}$ and $\{ue_1, \frac{1}{2\omega}uf_1\}$ are orthogonal hyperbolic pairs. Therefore, for any value of q, we see that V_1 is the sum of two non-degenerate, orthogonal subspaces, both of which contain non-zero singular vectors, so G_0^* has type $O_4^+(q)$ as claimed.

Note that the stabilizer $(G_0)_{V_1,V_2}$ in G_0 of the decomposition $V = V_1 \perp V_2$ contains $G_1 \circ G_2$ with $G_1 \cong \Omega_4^+(q)$ and $G_2 \cong \Omega_{n-4}^{\varepsilon}(q)$, while the corresponding stabilizer $(H_0)_{V_1,V_2}$ in H_0 contains $H_1 \circ H_2$ with $H_1 \cong \Omega_2^+(q^2)$ and $H_2 \cong \Omega_{n/2-2}^{\varepsilon'}(q^2)$. Moreover, by [14, Lemma 4.1.1], G_0^* and H_0^* are the respective full orthogonal groups. We distinguish several cases according to the value of q.

Case 1: *$p = 2$ and $q \ge 4$.* As previously remarked, we may assume $u \in \mathbb{F}_{q^2}$ satisfies $T(u) = 1$ and $u^{q+1} \ne 1$, whence $a = 1$ and $b \ne 0, 1$ in (6). We define

$$x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}(V_1),$$

with respect to the basis (e_1, f_1, ue_1, uf_1) of V_1. Noting that x stabilizes the subspaces in the decomposition $V_1 = \langle e_1, uf_1 \rangle \perp \langle e_1 + ue_1, f_1 \rangle$, it is easy to check that $Q(vx) = Q(v)$ for all $v \in V_1$ and so $x \in \mathrm{SO}(V_1) = \mathrm{SO}_4^+(q)$. Moreover, since x maps the totally singular 2-space $\langle e_1, e_1 + ue_1 \rangle$ to the trivially intersecting totally singular 2-space $\langle uf_1, f_1 \rangle$, it follows that $x \in \Omega_4^+(q)$ (see [14, p. 30]). Let $z := (x, 1) \in G_1 \times G_2$ and note that $z = z^{-1}$ and $z \in G_0$ (modulo scalars).

Let

$$A = \left(\begin{array}{c|c} A_0 & A_1 \\ \hline bA_1 & A_0 + A_1 \end{array} \right) \in H_0 \tag{23}$$

as in (7), and write

$$A_i = \begin{pmatrix} (A_i)_{11} & (A_i)_{12} \\ (A_i)_{21} & (A_i)_{22} \end{pmatrix} \tag{24}$$

where

$$(A_0)_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad (A_1)_{11} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

are 2×2 matrices. In addition, write

$$x = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad x^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \tag{25}$$

with respect to the basis (e_1, f_1, ue_1, uf_1) for V_1, where X_{11} and Y_{11} are blocks of size 2×2. Note that $x = x^{-1}$, so $X_{ij} = Y_{ij}$ for all i, j (this notation will be useful later on).

It is straightforward to verify that $z^{-1}Az$ is of the form (23) if and only if each of the following conditions holds:

(i) $(Y_{11}(A_0)_{11} + bY_{12}(A_1)_{11})(X_{11} + X_{12}) + (Y_{12}(A_0)_{11} + (Y_{11} + Y_{12})(A_1)_{11})(X_{21} + X_{22})$
$+(Y_{21}(A_0)_{11} + bY_{22}(A_1)_{11})X_{12} + (Y_{22}(A_0)_{11} + (Y_{21} + Y_{22})(A_1)_{11})X_{22} = 0$

(ii) $b(Y_{11}(A_0)_{11} + bY_{12}(A_1)_{11})X_{12} + b(Y_{12}(A_0)_{11} + (Y_{11} + Y_{12})(A_1)_{11})X_{22} + (Y_{21}(A_0)_{11} + bY_{22}(A_1)_{11})X_{11} + (Y_{22}(A_0)_{11} + (Y_{21} + Y_{22})(A_1)_{11})X_{21} = 0$
(iii) $(A_0)_{21}(X_{11} + X_{12} + X_{22}) + (A_1)_{21}(bX_{12} + X_{21}) = 0$
(iv) $(A_0)_{21}(bX_{12} + X_{21}) + (A_1)_{21}(bX_{11} + X_{21} + bX_{22}) = 0$
(v) $(Y_{11} + Y_{12} + Y_{22})(A_0)_{12} + (Y_{11} + (b+1)Y_{12} + Y_{21} + Y_{22})(A_1)_{12} = 0$
(vi) $(bY_{12} + Y_{21})(A_0)_{12} + b(Y_{11} + Y_{12} + Y_{22})(A_1)_{12} = 0.$

Since $b \neq 1$, conditions (i) and (ii) imply that

$$b_{11} = b_{22} = 0, \qquad a_{12} = (b+1)b_{12}, \qquad a_{21} = bb_{21},$$

while (iii)–(vi) indicate that each entry in the matrices $(A_0)_{12}$, $(A_1)_{12}$, $(A_0)_{21}$ and $(A_1)_{21}$ is zero. Therefore $H_0 \cap H_0^{z^{-1}} \leqslant (H_0)_{V_1,V_2}$. To compute the V_1-projection of $H_0 \cap H_0^{z^{-1}}$, note that

$$H_0^* = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix} \middle| \alpha \in \mathbb{F}_{q^2}^* \right\} \cong D_{2(q^2-1)} \tag{26}$$

with respect to the $\mathbb{F}_{q^2}$-basis (e_1, f_1) for V_1. By writing the elements of H_0^* in the form (23), it quickly follows that $H_0 \cap H_0^{z^{-1}}$ projects to the dihedral subgroup

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \lambda(u^2+1) \\ (\lambda(u^2+1))^{-1} & 0 \end{pmatrix} \middle| \lambda \in \mathbb{F}_q^* \right\}$$

of order $2(q-1)$. Hence $H_0 \cap H_0^{z^{-1}} < (H_0)_{V_1,V_2} < H_0$, and both inclusions are proper. Now we can finish the argument as in the proof of Proposition 5.5. The group $H \cap H^{z^{-1}}$ normalizes $H_0 \cap H_0^{z^{-1}}$, so it must normalize $H_2 \leqslant H_0 \cap H_0^{z^{-1}}$ and also it must normalize a dihedral $D_{2(q-1)}$ subgroup of H_1. Consequently, $H \cap H^{z^{-1}}$ must fix the subspace V_2 and its orthogonal complement V_1. Hence $H \cap H^{z^{-1}} \leqslant H_{V_1,V_2} < H$, where H_{V_1,V_2} is the H-stabilizer of the decomposition $V = V_1 \perp V_2$. The first inclusion is also proper, since $H_1 \leqslant H_{V_1,V_2}$ but $H_1 \not\leqslant H \cap H^{z^{-1}}$.

Case 2: $q = 2$. Here $a = b = 1$ in (6) and we set

$$x_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad x_0^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

with respect to the basis (e_1, f_1, ue_1, uf_1). Noting that x_0 exchanges the two components of the orthogonal decomposition $V_1 = \langle e_1, uf_1 \rangle \perp \langle e_1 + ue_1, f_1 \rangle$, it is easy to check that $x_0 \in \mathrm{SO}_4^+(2)$. Moreover, $x_0 \in \mathrm{SO}_4^+(2) \setminus \Omega_4^+(q)$ since x_0 maps the totally singular 2-space $\langle e_1, f_1 \rangle$ to the intersecting totally singular 2-space $\langle f_1, uf_1 \rangle$ (see [14, p. 30]).

First suppose $(n, \varepsilon) = (8, +)$ (and $q = 2$). Set

$$x = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix} \tag{27}$$

with respect to the ordered basis $(e_1, f_1, ue_1, uf_1, e_2, f_2, ue_2, uf_2)$. Then $x \in \Omega_8^+(2)$ and computation in GAP shows that $H_0 \cap H_0^{x^{-1}}$ is a proper subgroup of a Sylow 2-subgroup of H_0. Since G does not contain triality automorphisms, H is an extension of H_0 by a 2-group. In particular, $H \cap H^{x^{-1}}$ is a proper subgroup of a Sylow 2-group of H and thus G is not extremely primitive.

With the aid of Magma [3], it is straightforward to verify that there are no extremely primitive examples when $(n, q, \varepsilon) = (8, 2, -)$, so let us assume $n \geq 12$ and $q = 2$. Consider the orthogonal decomposition $V = V_3 \perp V_4$, where $V_3 = \langle e_i, f_i, ue_i, uf_i \mid i = 1, 2 \rangle$. Let $z \in G_0$ be the element fixing V_4 pointwise and acting on V_3 as the element x given in (27).

Let $A \in H_0$ be a matrix with blocks as in (23), and write A_i and the matrices x, x^{-1} defined above in block-matrix form as in (24) and (25), but with blocks $(A_i)_{11}, X_{11}, Y_{11}$ of size 4×4. Note that we obtain the blocks X_{ij} of x and the blocks Y_{ij} of x^{-1} by expressing x and x^{-1} in terms of the basis $(e_1, f_1, e_2, f_2, ue_1, uf_1, ue_2, uf_2)$, rather than the ordering $(e_1, f_1, ue_1, uf_1, e_2, f_2, ue_2, uf_2)$ used above in (27), so for example we have

$$X_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{21} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

As before, $z^{-1}Az$ is of the form (23) if and only if the equations labelled (i)–(vi) hold (with $b = 1$). It is straightforward to check that (iii) and (iv) imply that each entry in $(A_0)_{21}$ and $(A_1)_{21}$ is zero, and we obtain the same conclusion for $(A_0)_{12}$ and $(A_1)_{12}$ via (v) and (vi). Therefore $H_0 \cap H_0^{z^{-1}}$ is a subgroup of the H_0-stabilizer $(H_0)_{V_3,V_4}$ of the orthogonal decomposition

$V = V_3 \perp V_4$. Moreover, by our earlier analysis of the case $(n, q, \varepsilon) = (8, 2, +)$, we see that the V_3-projection of $H_0 \cap H_0^{z^{-1}}$ is a proper subgroup of a Sylow 2-subgroup of $\Omega_4^+(4)$. Therefore, the inclusions $H_0 \cap H_0^{z^{-1}} < (H_0)_{V_3,V_4} < H_0$ are proper and we conclude that $H \cap H^{z^{-1}} < H_{V_3,V_4} < H$, where H_{V_3,V_4} is the H-stabilizer of the decomposition $V = V_3 \perp V_4$.

Case 3: $p > 2$ *and* $q \geq 5$. In (6) we may assume $a = 0$ and $b = \omega$, where $\mathbb{F}_q^* = \langle \omega \rangle$. For now, let us assume $q \geq 5$, so $\omega \neq \pm 1$. Set

$$x = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\omega} \\ 0 & 0 & 2\omega & 0 \end{pmatrix} = x^{-1}$$

with respect to the basis (e_1, f_1, ue_1, uf_1) of V_1, and note that $xJx^T = J$ so $x \in \mathrm{SO}_4^+(q)$ (recall that J is defined in (22)). Now x respects the orthogonal decomposition $V_1 = \langle e_1, \frac{1}{2}f_1 \rangle \perp \langle ue_1, \frac{1}{2\omega}uf_1 \rangle$ and exchanges the singular vectors in these 2-dimensional G_0^*-modules, so $x \in \Omega_4^+(q)$. Set $z = (x, 1) \in G_1 \times G_2$ and note that $z \in G_0$ (modulo scalars).

Let $A \in H_0$ be a matrix with blocks as in (7), so

$$A = \left(\begin{array}{c|c} A_0 & A_1 \\ \hline \omega A_1 & A_0 \end{array}\right). \tag{28}$$

Express A_i, x and x^{-1} in block form as before (see (24) and (25)), where $(A_i)_{11}$, X_{11} and Y_{11} are 2×2 matrices. It is then straightforward to check that $z^{-1}Az$ has blocks as in (28) if and only if the following conditions hold:

(i)′ $(Y_{11}(A_0)_{11} + \omega Y_{12}(A_1)_{11})X_{11} + (Y_{12}(A_0)_{11} + Y_{11}(A_1)_{11})X_{21} = (Y_{21}(A_0)_{11} + \omega Y_{22}(A_1)_{11})X_{12} + (Y_{22}(A_0)_{11} + Y_{21}(A_1)_{11})X_{22}$
(ii)′ $\omega(Y_{11}(A_0)_{11} + \omega Y_{12}(A_1)_{11})X_{12} + \omega(Y_{12}(A_0)_{11} + Y_{11}(A_1)_{11})X_{22} = (Y_{21}(A_0)_{11} + \omega Y_{22}(A_1)_{11})X_{11} + (Y_{22}(A_0)_{11} + Y_{21}(A_1)_{11})X_{21}$
(iii)′ $(Y_{11} - Y_{22})(A_0)_{12} + (\omega Y_{12} - Y_{21})(A_1)_{12} = 0$
(iv)′ $(\omega Y_{12} - Y_{21})(A_0)_{12} + \omega(Y_{11} - Y_{22})(A_1)_{12} = 0$
(v)′ $(A_0)_{21}(X_{11} - X_{22}) + (A_1)_{21}(X_{21} - \omega X_{12}) = 0$
(vi)′ $(A_0)_{21}(\omega X_{12} - X_{21}) + \omega(A_1)_{21}(X_{22} - X_{11}) = 0$.

Since we are assuming $q \geq 5$ (and thus $\omega^2 \neq 1$), we deduce that

$$a_{12} = a_{21} = b_{11} = b_{22} = 0$$

from conditions (i)′ and (ii)′, where $(A_0)_{11} = (a_{ij})$ and $(A_1)_{11} = (b_{ij})$, while (iii)′–(vi)′ imply that each entry in $(A_0)_{12}$, $(A_0)_{21}$, $(A_1)_{12}$ and $(A_1)_{21}$ is zero. Therefore $H_0 \cap H_0^{z^{-1}}$ is contained in the H_0-stabilizer $(H_0)_{V_1,V_2}$ of the orthogonal decomposition $V = V_1 \perp V_2$. To compute the V_1-projection of $H_0 \cap H_0^{z^{-1}}$, first note that the elements of H_0^* are as in (26), and the matrix

$$\tilde{A} = \left(\begin{array}{c|c} (A_0)_{11} & (A_1)_{11} \\ \hline \omega(A_1)_{11} & (A_0)_{11} \end{array}\right)$$

satisfies the relation $\tilde{A}J\tilde{A}^T = J$. Therefore $a_{11}a_{22} + \omega b_{12}b_{21} = 1$ and $a_{11}b_{12} = a_{22}b_{21} = 0$, so either

$$a_{11} \neq 0, \quad A_0 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{11}^{-1} \end{pmatrix} \quad \text{and} \quad A_1 = 0,$$

or

$$b_{12} \neq 0, \quad A_0 = 0 \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & b_{12} \\ \frac{1}{\omega b_{12}} & 0 \end{pmatrix}.$$

It follows that $H_0 \cap H_0^{z^{-1}}$ projects to the subgroup

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & au \\ (au)^{-1} & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{F}_q^* \right\}$$

which has order $2(q-1)$, so

$$H_0 \cap H_0^{z^{-1}} < (H_0)_{V_1,V_2} < H_0$$

with proper inclusions. Therefore, by arguing as in the $p = 2$ case, we deduce that $H \cap H^{z^{-1}} < H_{V_1,V_2} < H$ and thus G is not extremely primitive.

Case 4: $q = 3$. Here $(a, b) = (0, -1)$ in (6). First suppose $(n, \varepsilon) = (8, +)$. With respect to the ordered basis $(e_1, f_1, e_2, f_2, ue_1, uf_1, ue_2, uf_2)$ we define

$$x = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad x^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

where

$$X_{11} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad X_{22} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \quad X_{12} = X_{21} = 0$$

and

$$Y_{11} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad Y_{22} = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad Y_{12} = Y_{21} = 0.$$

Since x fixes β we have $x \in \mathrm{SO}_8^+(3)$. In fact, it is easy to check that x belongs to the derived subgroup of $\mathrm{SO}_8^+(3)$, that is, $x \in \Omega_8^+(3)$. A straightforward Magma calculation reveals that $|H_0 \cap H_0^{x^{-1}}| = 288$ and we quickly deduce that $H \cap H^{x^{-1}}$ is not maximal in H. Similarly, a direct Magma calculation rules out any extremely primitive examples when $(n, q, \varepsilon) = (8, 3, -)$.

Now assume $n > 8$ (and $q = 3$). Consider the orthogonal decomposition $V = V_3 \perp V_4$, where $V_3 = \langle e_i, f_i, ue_i, uf_i \mid i = 1, 2\rangle$. Let $z \in G_0$ be the element fixing V_4 pointwise and acting on V_3 as the element x defined above in the case $(n, \varepsilon) = (8, +)$. In the usual way, if we consider an element $A \in H_0$ with blocks as in (28) and (24) (with $\omega = -1$ and $(A_i)_{11}$ of size 4×4) then $z^{-1}Az$ has the correct block structure if and only if conditions (i)′–(vi)′ hold. It is straightforward to check that (iii)′–(vi)′ imply that the entries in the matrices $(A_i)_{12}$ and $(A_i)_{21}$ are all zero, so

$$H_0 \cap H_0^{z^{-1}} \leqslant (H_0)_{V_3,V_4} \leqslant H_0,$$

where $(H_0)_{V_3,V_4}$ is the H_0-stabilizer of the decomposition $V = V_3 \perp V_4$. By considering the V_3-projection of $H_0 \cap H_0^{z^{-1}}$, and using the above analysis of the case $(n, q, \varepsilon) = (8, 3, +)$, we deduce that the first inclusion in this subgroup chain is proper. In addition, it is clear that the latter inclusion is also proper. We obtain $H \cap H^{z^{-1}} < H_{V_3,V_4} < H$ by the same argument as in all previous cases. □

Proposition 5.7. *Suppose $G_0 = \mathrm{PSp}_n(q)$ and $H \in \mathcal{C}_3$ is of type $\mathrm{GU}_{n/2}(q)$. Then G is not extremely primitive.*

Proof. According to [14, Proposition 4.3.7], H has a minimal normal subgroup which is cyclic of order $(q+1)/2$. Therefore, by Lemma 2.2, G is not extremely primitive. □

Proposition 5.8. *Suppose $G_0 = \mathrm{P}\Omega_n^\varepsilon(q)$ and $H \in \mathcal{C}_3$ is of type $\mathrm{GU}_{n/2}(q)$. Then G is not extremely primitive.*

Proof. According to [14, Proposition 4.3.18], either H has a nontrivial cyclic normal subgroup, or $(q, \varepsilon) = (3, -)$ and $n \equiv 2 \pmod 4$. In view of Lemma 2.2, we immediately reduce to the special case $(q, \varepsilon) = (3, -)$ with $n \equiv 2 \pmod 4$. Set $m = n/2$ and $H_0 = \mathrm{PSU}_m(3) = H \cap G_0$ and note that we may assume $n \geq 10$.

Let W be the natural H_0-module over $\mathbb{F}_9$ and let $\beta' : W \times W \to \mathbb{F}_9$ be a non-degenerate unitary form on W. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of W with respect to β' (see [14, Proposition 2.3.1]). Fix $u \in \mathbb{F}_9$ so that $u^2 = -1$ and $\{e_1, \ldots, e_m, ue_1, \ldots, ue_m\}$ is an $\mathbb{F}_3$-basis for the natural G_0-module V. For $v \in V$ we define $Q(v) = \beta'(v, v)$, so $Q : V \to \mathbb{F}_3$ is a non-degenerate quadratic form on V with associated bilinear form $\beta = T\beta'$ (see [14, Table 4.3.A]). Note that every $A \in H_0$ is of the form

$$A = \left(\begin{array}{c|c} A_0 & A_1 \\ \hline -A_1 & A_0 \end{array}\right) \tag{29}$$

(see (7)), with respect to the specific ordering $(e_1, \ldots, e_m, ue_1, \ldots, ue_m)$ of the above $\mathbb{F}_3$-basis for V. In addition, $J = -I_n$ is the matrix representing β and we calculate that a matrix A of the form (29) satisfies $AJA^T = J$ if and only if

$$A_0A_0^T + A_1A_1^T = I_m \quad \text{and} \quad A_1A_0^T = A_0A_1^T. \tag{30}$$

In addition, we note that the decomposition

$$V = \langle e_i, ue_i \mid 1 \leq i \leq 4\rangle \oplus \langle e_i, ue_i \mid 5 \leq i \leq m\rangle = V_1 \oplus V_2 \tag{31}$$

is orthogonal with respect to *both* β and β', and the restrictions of the respective forms to the two components V_1 and V_2 are non-degenerate.

Define

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and set

$$x_0 = \begin{pmatrix} I_m & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & I_{m-4} \end{pmatrix}$$

(once again with respect to the basis ordering $(e_1, \dots, e_m, ue_1, \dots, ue_m)$). Then $x_0 J x_0^T = J$ and $\det(x_0) = 1$, so $x_0 \in \mathrm{SO}_n^-(3)$ and $x := x_0^2 \in \Omega_n^-(3)$.

If we write A_0 and A_1 in block form as in (24), where $(A_i)_{11}$ has size 4×4, then it is straightforward to check that $x^{-1}Ax$ is of the form (29) if and only if each entry in the submatrices $(A_i)_{21}$ and $(A_i)_{12}$ is zero, and also $(A_0)_{11}$ and $(A_1)_{11}$ are of the form

$$(A_0)_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -a_{12} & a_{11} & -a_{14} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ -a_{32} & a_{31} & -a_{34} & a_{33} \end{pmatrix}, \qquad (A_1)_{11} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & -b_{11} & b_{14} & -b_{13} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{32} & -b_{31} & b_{34} & -b_{33} \end{pmatrix}.$$

Therefore

$$H_0 \cap H_0^{x^{-1}} \leqslant (H_0)_{V_1,V_2} < H_0,$$

where $(H_0)_{V_1,V_2}$ is the H_0-stabilizer of the decomposition (31). Moreover, $(A_0)_{11}$ and $(A_1)_{11}$ have the above form, and also satisfy the conditions in (30). More precisely, computation in GAP shows that the V_1-projection of $H_0 \cap H_0^{x^{-1}}$ is isomorphic to $\mathrm{Sp}_4(3)$, whence $H_0 \cap H_0^{x^{-1}} < (H_0)_{V_1,V_2}$ is a proper inclusion. Finally, the usual argument now implies that $H \cap H^{x^{-1}} < H_{V_1,V_2} < H$ and we conclude that G is not extremely primitive. $\square$

6. Tensor product subgroups

Here we deal with the stabilizers of tensor product decompositions of V, which comprise the $\mathcal{C}_4$ and $\mathcal{C}_7$ subgroup collections. The specific cases we have to consider are listed in [14, Tables 4.4.A and 4.7.A].

Proposition 6.1. *Let G be an almost simple primitive classical group with point stabilizer $H \in \mathcal{C}_4 \cup \mathcal{C}_7$. Then G is not extremely primitive.*

Proof. According to [5, Propositions 6.1 and 6.4], either $b(G) = 2$, or $G_0 = \mathrm{P}\Omega_8^+(q)$ and H is a $\mathcal{C}_4$-subgroup of type $\mathrm{Sp}_4(q) \otimes \mathrm{Sp}_2(q)$. If $b(G) = 2$ then G is not extremely primitive by Lemma 2.1, while in the remaining case we observe that the socle of H is not a product of isomorphic simple groups. The result follows. $\square$

7. Subfield subgroups

Let H be a maximal subgroup of G in Aschbacher's $\mathcal{C}_5$ collection. Here H corresponds to a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$ such that $q = q_0^r$ for some prime r. The various possibilities for G and H are listed in [14, Table 4.5.A].

Proposition 7.1. *If $r \geq 3$ then $b(G) = 2$ and thus G is not extremely primitive.*

Proof. This follows immediately from [5, Proposition 5.1] and Lemma 2.1. $\square$

For the remainder of this section we may assume H corresponds to an index-two subfield of $\mathbb{F}_q$. The next lemma provides a useful description of $H \cap H^x$.

Lemma 7.2. *Let $\bar{G}$ be an algebraic group over the algebraic closure of $\mathbb{F}_q$. Let σ be a Frobenius morphism of $\bar{G}$ and set $G = \bar{G}_{\sigma^2}$ and $H = \bar{G}_\sigma$, where*

$$\bar{G}_{\sigma^i} = \{x \in \bar{G} \mid \sigma^i(x) = x\}.$$

Then $H \cap H^x = C_H(x^{-1}\sigma(x))$ for all $x \in G$.

Proof. First observe that $y \in H \cap H^x$ if and only if $y \in H$ and $Hxy = Hx$. Since $H = \bar{G}_\sigma$, the latter condition is equivalent to $\sigma(xyx^{-1}) = xyx^{-1}$. Further, using the fact that σ is a group homomorphism and $\sigma(y) = y$, we quickly deduce that $y \in H$ and $Hxy = Hx$ if and only if $y \in C_H(x^{-1}\sigma(x))$. The result follows. $\square$

Proposition 7.3. *Let G be an almost simple primitive classical group with socle G_0 and point stabilizer H, where $H \in \mathcal{C}_5$ is one of the following:*

	G_0	Type of H	Conditions
(i)	$\mathrm{PSL}_n(q)$	$\mathrm{GL}_n(q_0)$	$q = q_0^2$
(ii)	$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_n(q_0)$	$n \geq 4$, $q = q_0^2$
(iii)	$\mathrm{P}\Omega_n^\varepsilon(q)$	$\mathrm{O}_n^{\varepsilon'}(q_0)$	$n \geq 7$, $q = q_0^2$, $\varepsilon = +$ *if n even*

Then G is not extremely primitive.

Proof. *Case* (i) *with no graph automorphisms:* Let $\bar{G}$ be the ambient simple algebraic group $\mathrm{PSL}_n(K)$, where K is the algebraic closure of $\mathbb{F}_q$, and let σ be a Frobenius morphism of $\bar{G}$ such that $(\bar{G}_{\sigma^2})' = G_0$ and $(\bar{G}_\sigma)' = H_0 = \mathrm{PSL}_n(q_0)$. Note that $H_0 \leqslant H \cap G_0$. Let V be the natural G_0-module (where we consider the action of $\mathrm{SL}_n(q)$ rather than $\mathrm{PSL}_n(q)$) and fix a basis $(v_1, \ldots, v_n)$ for V. Without loss of generality, we may assume that σ is the standard involutory field automorphism of G_0 with respect to this fixed basis, so $\sigma : (a_{ij}) \mapsto (a_{ij}^{q_0})$. If $n = 2$ and $q_0 \leq 3$ then using Magma it is easy to check that G is not extremely primitive, so we may assume H_0 is simple.

Write $\mathbb{F}_q^* = \langle \omega \rangle$ and set

$$x = \left(\begin{array}{cc|c} 1 & \omega & \\ 0 & 1 & \\ \hline & & I_{n-2} \end{array}\right) \in G_0, \qquad y = x^{-1}\sigma(x) = \left(\begin{array}{cc|c} 1 & \omega^{q_0} - \omega & \\ 0 & 1 & \\ \hline & & I_{n-2} \end{array}\right),$$

so $H_0 \cap H_0^x = C_{H_0}(y)$ by Lemma 7.2. We calculate that $C_{H_0}(y)$ is the set of matrices in H_0 with first column $(\lambda, 0, \ldots, 0)^T$ and second row $(0, \lambda, 0, \ldots, 0)$ for some $\lambda \in \mathbb{F}_{q_0}^*$ (and $\lambda = 1$ if $n = 2$). Therefore

$$S \leqslant H_0 \cap H_0^x \leqslant (H_0)_{U,W} \leqslant (H_0)_U, \tag{32}$$

where $U = \langle v_2 \rangle$, $W = \langle v_2, \ldots, v_n \rangle$ and S is a Sylow p-subgroup of H_0. We calculate that

$$|(H_0)_U : (H_0)_{U,W}| = \frac{q_0^{n-1} - 1}{q_0 - 1} \quad \text{and} \quad |(H_0)_{U,W} : H_0 \cap H_0^x| = q_0 - 1$$

and thus $H_0 \cap H_0^x < (H_0)_U$ (recall that we are assuming $q_0 \geq 4$ when $n = 2$). Now if G does not contain any graph automorphisms then Lemma 2.3 implies that $H \cap H^x$ is not a maximal subgroup of H, whence G is not extremely primitive.

Case (i) *with graph automorphisms:* Assume that $n \geq 3$ and G contains graph automorphisms. Set $\tilde{G} = G \cap \mathrm{P}\Gamma\mathrm{L}_n(q)$ and $\tilde{H} = H \cap \tilde{G}$; and set $L := H \cap H^x$, $\tilde{L} := L \cap \tilde{G} = \tilde{H} \cap \tilde{H}^x$, and $L_0 := L \cap G_0 = H_0 \cap H_0^x$. We use some arguments from the proof of Lemma 2.3. We refer to an unordered subspace pair $\{U', W'\}$ of V, with $\dim U' = 1$, $\dim W' = n-1$, and $U' \subseteq W'$, as a *flag*; in particular the pair $\{U, W\}$ above is a flag stabilized by L_0.

As we showed above, the group L_0 contains a Sylow p-subgroup S of H_0, and so we have $L = L_0 N_L(S)$. Thus the subgroup $H_0 N_L(S)$ of H contains L with index $|H_0 : L_0|/|N_{H_0}(S) : N_{L_0}(S)|$. Now $N_{H_0}(S)$ is a Borel subgroup of H_0 contained in L_0, and hence $N_{H_0}(S) = N_{L_0}(S)$, so $|H_0 N_L(S) : L| = |H_0 : L_0| > 1$. In particular, if $H_0 N_L(S) \neq H$ then L is not maximal in H and G is not extremely primitive. Hence we may assume that $H = H_0 N_L(S)$. Since H is maximal in G, we have $G = G_0 H = G_0 N_L(S)$.

Thus, for some graph automorphism τ, we have $L = \langle \tilde{L}, \tau \rangle$, $H = \langle \tilde{H}, \tau \rangle$ and $G = \langle \tilde{G}, \tau \rangle$. Since τ normalizes $\tilde{L}$ and G_0 it follows that τ normalizes $\tilde{L} \cap G_0 = L_0$. Note that, since τ interchanges stabilizers of 1-subspaces and stabilizers of $(n-1)$-subspaces, reversing inclusion, τ induces an action on flags. Before proceeding we observe that our arguments above show that $L_0 = C_{H_0}(y)$ and $(H_0)_{U,W}$ induce the same action on W, and in particular L_0 fixes no $(n-2)$-subspace of W; also L_0 fixes a unique 1-subspace of V, namely U. It follows that $\{U, W\}$ is the unique flag fixed by L_0, since if $\{U', W'\}$ is another flag fixed by L_0, with $\dim U' = 1$, $\dim W' = n-1$, and $U' \subseteq W'$, then $W' = W$ (since otherwise $W' \cap W$ would be an $(n-2)$-subspace of W fixed by L_0), and $U' = U$ (since otherwise L_0 would fix two 1-subspaces). Then, since L_0 is normal in L, the subgroup L fixes $\{U, W\}$, and therefore also $\tilde{L}$ and τ fix $\{U, W\}$.

Hence $L \leqslant H_{\{U,W\}} < H$. The second inclusion is clearly proper, and we examine the first more closely. Since $H = H_0 N_L(S)$ and $N_L(S)$ fixes $\{U, W\}$, we have $H_{\{U,W\}} = (H_0)_{\{U,W\}} N_L(S)$. Since also $L = L_0 N_L(S)$, this implies that

$$|H_{\{U,W\}}| = \frac{|(H_0)_{\{U,W\}}| \cdot |N_L(S)|}{|N_{L_0}(S)|}, \qquad |L| = \frac{|L_0| \cdot |N_L(S)|}{|N_{L_0}(S)|}$$

and hence $|H_{\{U,W\}} : L| = |(H_0)_{\{U,W\}} : L_0|$. Since $n \geq 3$, we have $(H_0)_{\{U,W\}} = (H_0)_{U,W}$, and we showed above that $|(H_0)_{U,W} : L_0| = q_0 - 1$. Thus provided $q_0 \geq 3$, L is not maximal in H and so G is not extremely primitive.

We are left with the case $q_0 = 2$. Here H acts primitively on the above suborbit, so we examine a different suborbit. Note that in this final case, since we have $Z(H) = 1$ and H maximal in G, G does not contain any diagonal automorphisms, or any involutory field automorphisms. Thus $\tilde{G} = G_0$ and $G = G_0.2$. As above let $\mathbb{F}_4^* = \langle \omega \rangle$. We re-define

$$x = \left(\begin{array}{c|cc} A & & \\ \hline & \omega^2 & 0 \\ & \omega^2 & \omega \end{array}\right) \in G_0,$$

where $A \in \mathrm{SL}_{n-2}(2)$ has all diagonal entries equal to 1, all super-diagonal entries equal to ω, and all other entries 0. As before we define $y = x^{-1}\sigma(x)$ and we have $H_0 \cap H_0^x = C_{H_0}(y)$ by Lemma 7.2. We calculate that $C_{H_0}(y)$ consists of all upper-triangular matrices of the form $[C, I_2]$ where $C \in \mathrm{SL}_{n-2}(2)$ is upper-triangular such that, on each diagonal above the main diagonal, the entries are constant (and equal to either 0 or 1). This implies that $|H_0 \cap H_0^x| = |C_{H_0}(y)| = 2^{n-3}$, and hence that $|H \cap H^x| = 2^{n-3}$ or 2^{n-2}, and in either case $H \cap H^x$ is not maximal in H. Thus G is not extremely primitive.

Case (ii): Let σ be a Frobenius morphism of $\bar{G} = \mathrm{PSp}_n(K)$ such that $(\bar{G}_{\sigma^2})' = G_0$ and $(\bar{G}_\sigma)' = H_0 = \mathrm{PSp}_n(q_0)$. Let $\mathbb{F}_q^* = \langle \omega \rangle$, $m = n/2$ and fix a standard symplectic basis $(e_1, f_1, \ldots, e_m, f_m)$ for V. As in (i), we may assume σ is the standard involutory field automorphism with respect to this basis. Set $V_1 = \langle e_1, f_1\rangle$ and $V_2 = \langle e_2, f_2, \ldots, e_m, f_m\rangle$, so $V = V_1 \perp V_2$ is an orthogonal decomposition. According to [14, Proposition 4.1.3], the H_0-stabilizer of this decomposition is $H_1 \circ H_2$, where $H_1 \cong \mathrm{Sp}_2(q_0)$ and $H_2 \cong \mathrm{Sp}_{n-2}(q_0)$.

Let $x = [\omega, \omega^{-1}, I_{n-2}] \in G_0$, so $y = x^{-1}\sigma(x) = [\omega^{q_0-1}, \omega^{1-q_0}, I_{n-2}]$. Then

$$Z_{q_0-1} \times H_2 \leqslant C_{H_0}(y) = H_0 \cap H_0^x \leqslant M_0 \times H_2 < (H_0)_{V_1,V_2}, \tag{33}$$

where M_0 is a $\mathcal{C}_2$-subgroup of $H_1 \cong \mathrm{SL}_2(q_0)$. Now $H \cap H^x$ normalizes $H_0 \cap H_0^x$, so it also normalizes H_2 (and M_0 if $q_0 > 2$). Suppose first that, if $n = 4$, then G involves no graph–field automorphisms. Then $H \cap H^x$ fixes the decomposition $V = V_1 \perp V_2$. In other words,

$$H \cap H^x \leqslant H_{V_1,V_2} < H$$

where H_{V_1,V_2} is the H-stabilizer of the subspaces V_1 and V_2. Moreover, the first inclusion is also proper since $H_1 \leqslant H_{V_1,V_2}$ but $H_1 \not\leqslant H \cap H^x$. Thus we may assume that $n = 4$ and G contains graph–field automorphisms. The case $q_0 = 2$ is easily checked using Magma, so let us assume $q_0 \geq 4$.

Suppose that G is extremely primitive. Then G_0 acts transitively on the orbital $(\alpha, \beta)^G$, where $H = G_\alpha$, and $H^x = G_\beta$, and hence $G = G_0(H \cap H^x)$. It follows that $H \cap H^x$ also contains a graph–field automorphism, τ say. Since $q_0 \geq 4$, then by (33), $H_2 \cong \mathrm{Sp}_2(q_0)$ is a characteristic subgroup of $H_0 \cap H_0^x$, and hence is normalized by τ. Since τ normalizes H_0, τ also normalizes $C_{H_0}(H_2) = H_1$, and hence τ normalizes $H_1 \times H_2 = (H_0)_{V_1,V_2}$ and its normalizer in H_0. This is a contradiction since τ does not leave invariant this conjugacy class of maximal $\mathcal{C}_2$-subgroups of H_0 (see [1, (14.1)]).

Case (iii) *with no triality automorphisms:* Let σ be a suitable Frobenius morphism of $\bar{G} = \mathrm{PSO}_n(K)$ such that $(\bar{G}_{\sigma^2})' = G_0$ and $(\bar{G}_\sigma)' = H_0 = \mathrm{P}\Omega_n^{\varepsilon'}(q_0)$. Let $\{e_1, f_1, e_2, f_2, \ldots\}$ be a standard orthogonal basis for V with respect to the quadratic form defining G, where $V_1 = \langle e_1, f_1, e_2, f_2\rangle$ is a non-degenerate 4-space of plus type. Without loss of generality, we may assume σ acts as a standard field automorphism on V_1. Let $V_2 = V_1^\perp$ and note that the H_0-stabilizer of the orthogonal decomposition $V = V_1 \perp V_2$ is a central product $H_1 \circ H_2$, where H_1 is of type $O_4^+(q_0)$ and H_2 is of type $O_{n-4}^{\varepsilon'}(q_0)$ (the precise structure is given in [14, Proposition 4.1.6]). As before, write $\mathbb{F}_q^* = \langle \omega \rangle$.

To begin with, let us assume G does not contain a triality automorphism when $n = 8$. Let $x \in \mathrm{SO}_n^\varepsilon(q)$ be the diagonal matrix $x = [\omega I_2, \omega^{-1} I_2, I_{n-4}]$ with respect to the specific basis ordering $(e_1, e_2, f_1, f_2, \ldots)$. By [14, Lemma 4.1.1(iv)] we have $x \in G_0$ (modulo scalars). Let $y = x^{-1}\sigma(x) = [\omega^{q_0-1} I_2, \omega^{1-q_0} I_2, I_{n-4}]$ and define $U = \langle e_1, e_2\rangle$, $W = \langle f_1, f_2\rangle$. Then

$$L_0 \times H_2 \leqslant C_{H_0}(y) = H_0 \cap H_0^x \leqslant M_0 \times H_2,$$

where $L_0 = (H_1)_{U,W}$ and $M_0 = (H_1)_{\{U,W\}}$. In the usual manner we deduce that

$$H \cap H^x < H_{V_1,V_2} < H$$

and thus G is not extremely primitive.

Case (iii) *with triality automorphisms:* To complete the proof, let us assume $(n, \varepsilon) = (8, +)$ and G contains a triality automorphism. Set $H_0 = \mathrm{P}\Omega_8^+(q_0)$. Then according to [14, Proposition 4.5.10 and Table 2.1.D on p. 19] we have $H \cap G_0 = H_0.[c]$, where $c = 1$ if $p = 2$, otherwise $c = 4$ (if $p \neq 2$ then [14, Proposition 2.5.10(i)] implies that the discriminant of H_0 is a square in $\mathbb{F}_{q_0}$). We may assume that H is almost simple with socle H_0 (note that $Z(H) \neq 1$ if G contains an involutory field automorphism), and that H contains a triality automorphism of H_0.

Let x be the block-diagonal matrix $x = [I_2, \omega^2, \omega^{-2}, A, B]$ with respect to the basis $(e_1, f_1, e_2, f_2, e_3, e_4, f_3, f_4)$, where

$$A = \begin{pmatrix} \omega & \omega^2 \\ 0 & \omega \end{pmatrix}, \qquad B = \begin{pmatrix} \omega^{-1} & 0 \\ -1 & \omega^{-1} \end{pmatrix} = A^{-T},$$

and observe that $x \in G_0$ (see [14, Lemma 4.1.1]). Set $L = H \cap H^x$ and $L_0 = L \cap H_0$. Note that $L_0 = H_0 \cap (H \cap G_0)^x$ and

$$|H_0 \cap H_0^x| \leq |L_0| \leq |H \cap G_0 : H_0| \cdot |H_0 \cap H_0^x| = c|H_0 \cap H_0^x|.$$

As before, we have $H_0 \cap H_0^x = C_{H_0}(y)$, where $y = x^{-1}\sigma(x)$ is the block-diagonal matrix $y = [I_2, \omega^{2(q_0-1)}, \omega^{-2(q_0-1)}, C, C^{-T}]$ with

$$C = \begin{pmatrix} \omega^{q_0-1} & \omega^{2q_0-1} - \omega^{q_0} \\ 0 & \omega^{q_0-1} \end{pmatrix}.$$

Now, if G is extremely primitive then L is a maximal subgroup of H. In particular, L must be one of the subgroups listed in [12, Table III], with $|L_0|$ recorded in the second column of this table.

First assume $q_0 = 2$. With the aid of Magma we calculate that $L_0 = H_0 \cap H_0^x \cong D_8$. However, [12, Table III] indicates that there is no maximal subgroup M of H with $|M \cap H_0| = 8$, so L is not a maximal subgroup of H and thus G is not extremely primitive. Similarly, if $q_0 = 3$ then $H_0 \cap H_0^x \cong Z_6$ and the same conclusion follows.

Finally, suppose $q_0 \geq 4$. It is straightforward to check that $C_{H_0}(y)$ is the set of block-diagonal matrices in H_0 of the form $[D, \lambda, \lambda^{-1}, E, E^{-T}]$, where $D \in \mathrm{SO}_2^+(q_0)$, $\lambda \in \mathbb{F}_{q_0}^*$ and

$$E \in \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a \in \mathbb{F}_{q_0}^*, b \in \mathbb{F}_{q_0} \right\} < \mathrm{GL}_2(q_0).$$

Therefore $|H_0 \cap H_0^x| = \frac{1}{d}q_0(q_0-1)^3$, where $d = (2, q-1)$, and by inspecting [12, Table III], as before, we deduce that G is not extremely primitive. $\square$

Proposition 7.4. *Suppose $G_0 = \mathrm{PSU}_n(q)$ and $H \in \mathcal{C}_5$ is of type $\mathrm{Sp}_n(q)$, where n is even and $n \geq 4$. Then G is not extremely primitive.*

Proof. If G contains a graph automorphism of G_0 then $Z(H) = Z_2$ is nontrivial, and thus G is not extremely primitive by Lemma 2.2(i). For the remainder we may assume otherwise. Write $n = 2m$ and let $\mathcal{B} = \{e_1, f_1, \dots, e_m, f_m\}$ be a standard symplectic basis for an n-dimensional vector space W over $\mathbb{F}_q$ equipped with a symplectic form β'. Fix $u \in \mathbb{F}_{q^2}^*$ such that $u^q = -u$ and set

$$V = \{(a+bu)w \mid a, b \in \mathbb{F}_q,\ w \in W\}$$

if q is odd, and

$$V = \{aw \mid a \in \mathbb{F}_{q^2},\ w \in W\}$$

if q is even, so V is an n-dimensional vector space over $\mathbb{F}_{q^2}$, with basis $\mathcal{B}$. Define a form $\beta : V \times V \to \mathbb{F}_{q^2}$ by

$$\beta((a_1+b_1u)v, (a_2+b_2u)w) = (a_1+b_1u)(a_2-b_2u)\beta'(v,w)u.$$

Then β is a non-degenerate unitary form on V (see [14, p. 143]) and

$$J = \begin{pmatrix} & I_m \\ -I_m & \end{pmatrix}, \qquad K = \begin{pmatrix} & uI_m \\ -uI_m & \end{pmatrix}$$

are the matrices of the forms β' and β, respectively, expressed in terms of the ordered basis $(e_1, \dots, e_m, f_1, \dots, f_m)$. Set $H_0 = \mathrm{PSp}_n(q) \leqslant H \cap G_0$. Without loss of generality, we may assume that V is the natural G_0-module and that G_0 fixes β and H_0 fixes β'. In other words, modulo scalars we have

$$G_0 = \{x \in \mathrm{SL}_n(q^2) \mid xK\bar{x}^T = K\}$$
$$H_0 = \{x \in \mathrm{SL}_n(q^2) \mid xK\bar{x}^T = K \text{ and } xJx^T = J\},$$

where $\bar{x} = (x_{ij}^q)$ for $x = (x_{ij}) \in \mathrm{SL}_n(q^2)$. Also note that if $x \in G_0$ then H_0^x is the stabilizer (in G_0) of the symplectic form corresponding to the asymmetric matrix $x^{-1}Jx^{-T}$. In particular, we claim that

$$H_0 \cap H_0^x = C_{H_0}(y) \quad \text{where } y = x^{-1}Jx^{-T}J^{-1}. \tag{34}$$

To see this, note that $z \in H_0 \cap H_0^x$ if and only if $zJz^T = J$ and $z(x^{-1}Jx^{-T})z^T = x^{-1}Jx^{-T}$. Here the former condition is equivalent to $z^T = J^{-1}z^{-1}J$, so $z \in H_0 \cap H_0^x$ if and only if

$$x^{-1}Jx^{-T} = z(x^{-1}Jx^{-T})J^{-1}z^{-1}J,$$

which is equivalent to the condition $z \in C_{H_0}(y)$.

Write $\mathbb{F}_{q^2}^* = \langle \omega \rangle$ and set $V_1 = \langle e_1, f_1, e_2, f_2 \rangle$ and $V_2 = \langle e_3, f_3, \dots, e_m, f_m \rangle$, so $V = V_1 \perp V_2$ is an orthogonal decomposition with respect to the symplectic form β'. By [14, Proposition 4.1.3], the H_0-stabilizer of this decomposition is a central product $H_1 \circ H_2$, where $H_1 \cong \mathrm{Sp}_4(q)$ and $H_2 \cong \mathrm{Sp}_{n-4}(q)$.

First let us assume q is even. Fix the basis ordering $(e_1, f_1, \dots, e_m, f_m)$ and define $x = [\omega^{q-1}I_2, \omega^{1-q}I_2, I_{n-4}] \in G_0$ and

$$y = x^{-1}Jx^{-T}J^{-1} = [\omega^{2(1-q)}I_2, \omega^{2(q-1)}I_2, I_{n-4}].$$

If $n \geq 6$ then

$$\mathrm{Sp}_2(q) \times \mathrm{Sp}_2(q) \times H_2 = C_{H_0}(y) = H_0 \cap H_0^x < (H_0)_{V_1,V_2}$$

and the usual argument implies that $H \cap H^x < H_{V_1,V_2} < H$. Similarly, if $n = 4$ then

$$\mathrm{Sp}_2(q) \times \mathrm{Sp}_2(q) = C_{H_0}(y) = H_0 \cap H_0^x = (H_0)_{U_1,U_2} < (H_0)_{\{U_1,U_2\}},$$

where $U_1 = \langle e_1, f_1 \rangle$ and $U_2 = \langle e_2, f_2 \rangle$. Therefore

$$H \cap H^x = H_{U_1,U_2} < H_{\{U_1,U_2\}} < H$$

and once again we conclude that G is not extremely primitive.

A similar argument applies when q is odd. Here we set

$$x = \left(\begin{array}{cccc|c} \omega^i & 0 & & & \\ \omega^j & \omega^j & & & \\ & & \omega^{-i} & 0 & \\ & & -\omega^{-i} & \omega^{-j} & \\ \hline & & & & I_{n-4} \end{array}\right)$$

in terms of the basis $(e_1, f_1, \ldots, e_m, f_m)$, where $(i, j) = (q-1, q-1)$ if $q \geq 5$, and $(i, j) = (1, 5)$ when $q = 3$. This choice of i and j implies that $xK\bar{x}^T = K$, so $x \in G_0$. Now $y = x^{-1}Jx^{-T}J^{-1}$ is the diagonal matrix $[\omega^{-i-j}I_2, \omega^{i+j}I_2, I_{n-4}]$, and we note that $\omega^{-i-j} \neq \omega^{i+j}$. We can now complete the argument as in the q even case. □

Proposition 7.5. *Suppose $G_0 = \mathrm{PSU}_n(q)$ and $H \in \mathcal{C}_5$ is of type $O_n^{\varepsilon}(q)$, where q is odd and $n \geq 3$. Then G is not extremely primitive.*

Proof. As in the proof of the previous proposition, we may assume G does not contain any graph automorphisms. Set $H_0 = \mathrm{PSO}_n^{\varepsilon}(q)$ and assume $n \geq 5$ for now. By [14, Proposition 4.5.4] we have $H_0 \leqslant H \cap G_0$. Let V be the natural G_0-module and let $\mathcal{B} = \{e_1, f_1, e_2, f_2, \ldots\}$ be a basis for V with respect to a non-degenerate unitary form β, where $\beta(e_1, e_2) = \beta(f_1, f_2) = \beta(e_1, f_2) = \beta(e_2, f_1) = 0$ and $\beta(e_i, f_i) = 1$ (see [14, Proposition 2.3.2]). Moreover, we may choose the basis $\mathcal{B}$ and a specific ordering $(e_1, f_1, e_2, f_2, \ldots)$ so that

$$J = \left(\begin{array}{cccc|c} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ \hline & & & & * \end{array}\right) \tag{35}$$

is a symmetric matrix representing β, and modulo scalars we have

$$G_0 = \{x \in \mathrm{SL}_n(q^2) \mid xJ\bar{x}^T = J\}$$
$$H_0 = \{x \in \mathrm{SL}_n(q^2) \mid xJ\bar{x}^T = J \text{ and } xJx^T = J\}.$$

Set $\mathbb{F}_{q^2}^* = \langle \omega \rangle$, $V_1 = \langle e_1, f_1, e_2, f_2 \rangle$ and $V_2 = V_1^{\perp}$. Note that the H_0-stabilizer of the orthogonal decomposition $V = V_1 \perp V_2$ is a central product of the form $H_1 \circ H_2$, where H_1 is of type $O_4^+(q)$ and H_2 is of type $O_{n-4}^{\varepsilon}(q)$. Also define $U_1 = \langle e_1, f_1 \rangle$ and $U_2 = \langle e_2, f_2 \rangle$. As in the proof of the previous proposition, we note that (34) holds for all $x \in G_0$.

Let $x = [\omega^{q-1}I_2, \omega^{1-q}I_2, I_{n-4}] \in G_0$ (with respect to the above basis) and define

$$y = x^{-1}Jx^{-T}J^{-1} = [\omega^{2(1-q)}I_2, \omega^{2(q-1)}I_2, I_{n-4}].$$

Then

$$L_0 \times H_2 = C_{H_0}(y) = H_0 \cap H_0^x < (H_0)_{V_1,V_2},$$

where $L_0 = (H_1)_{U_1,U_2}$ is a subgroup of H_1 of type $O_2^+(q) \times O_2^+(q)$. The usual argument now yields

$$H \cap H^x < H_{V_1,V_2} < H$$

and thus G is not extremely primitive.

To complete the proof, let us assume $n \leq 4$. If $q = 3$ then the result is easily checked using Magma, so we will assume $q \geq 5$. First suppose $(n, \varepsilon) = (4, +)$. Define $x \in G_0$ and $y = x^{-1}Jx^{-T}J^{-1}$ as in the previous paragraph. Then

$$L_0 = C_{H_0}(y) = H_0 \cap H_0^x < (H_0)_{\{U_1,U_2\}},$$

where L_0 is defined as before, and $(H_0)_{\{U_1,U_2\}}$ is a $\mathcal{C}_2$-subgroup of H_0 of type $O_2^+(q) \wr S_2$. It follows that $H \cap H^x < H_{\{U_1,U_2\}} < H$.

Now assume $(n, \varepsilon) = (4, -)$. Let $\{v_1, v_2, v_3, v_4\}$ be an orthonormal basis for V with respect to β (see [14, Proposition 2.3.1]) and consider the basis $\mathcal{B} = \{\omega v_1, v_2, v_3, v_4\}$. Now the diagonal matrix $J = [\omega^{q+1}, I_3]$ represents β with respect to $\mathcal{B}$, and we also note that $\det(J) = \omega^{q+1}$ is a nonsquare element of $\mathbb{F}_q$, so H_0 is of type $O_4^-(q)$ as desired. Let x be the diagonal matrix $x = [I_2, \omega^{q-1}, \omega^{1-q}]$. Then $x \in G_0$ since $xJ\bar{x}^T = J$, and we have $H_0 \cap H_0^x = C_{H_0}(y)$, where

$$y = x^{-1}Jx^{-T}J^{-1} = [I_2, \omega^{2-2q}, \omega^{2q-2}].$$

It is easy to check that each $z \in C_{H_0}(y)$ is a block-diagonal matrix of the form $z = [X, a, b]$, where $X \in \mathrm{GL}_2(q)$ and $a^2 = b^2 = 1$. As a consequence, we deduce that $H \cap H^x < L < H$, where L is the H-stabilizer of the orthogonal decomposition $V = \langle \omega v_1, v_2 \rangle \oplus \langle v_3, v_4 \rangle$.

Finally, suppose $n = 3$. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis for V (with respect to the unitary form β) and set $x = [1, \omega^{q-1}, \omega^{1-q}] \in G_0$ with respect to the ordered basis (v_1, v_2, v_3). Then $H_0 \cap H_0^x = C_{H_0}(y)$, where $y = x^{-1}x^{-T} =$

$[1, \omega^{2-2q}, \omega^{2q-2}]$, and we deduce that

$$H \cap H^x = H_{\langle v_1\rangle,\langle v_2\rangle,\langle v_3\rangle} < H_{\{\langle v_1\rangle,\langle v_2\rangle,\langle v_3\rangle\}} < H.$$

The result follows. □

8. Symplectic-type normalizers

Let $r \neq p$ be a prime. Recall that an r-group R is *extraspecial* if $Z(R) = \Phi(R) = R' = Z_r$, where $\Phi(R)$ and R' denote the Frattini subgroup and derived group of R, respectively. Further, an extraspecial group R is of *symplectic-type* if every characteristic abelian subgroup of R is cyclic. The members of Aschbacher's $\mathcal{C}_6$ collection are the normalizers of certain absolutely irreducible symplectic-type r-groups; the various cases to be considered are listed in [14, Table 4.6.B], and we refer the reader to [14, Section 4.6] for further details on the structure of these subgroups.

Proposition 8.1. *Let G be an almost simple primitive classical group with socle G_0 and point stabilizer $H \in \mathcal{C}_6$. Then G is extremely primitive if and only if $G_0 = \mathrm{PSL}_2(5)$ and H is of type $2^2.O_2^-(2)$. (This group is permutationally isomorphic to $\mathrm{PSL}_2(4)$ or $\mathrm{PSL}_2(4).2$ on the cosets of P_1, as in line 1 of Table 1.)*

Proof. According to [5, Proposition 7.1], either $b(G) = 2$, or the action of G is permutation isomorphic to a subspace action, or (G, H) is one of the following cases:

	G_0	Type of H	Conditions
(i)	$\mathrm{PSL}_2(5)$	$2^2.O_2^-(2)$	
(ii)	$\mathrm{PSU}_4(3)$	$2^4.\mathrm{Sp}_4(2)$	
(iii)	$\mathrm{PSp}_4(5)$	$2^4.O_4^-(2)$	$G = G_0.2$
(iv)	$\mathrm{P}\Omega_8^+(3)$	$2^6.O_6^+(2)$	$G = G_0.2 < \mathrm{Inndiag}(G_0),\ G \neq \mathrm{PSO}_8^+(3)$

In view of Lemma 2.1 and our work in Section 3 on reducible subgroups, it remains to deal with the cases (i)–(iv) listed above. In (i) the action of G is isomorphic to the natural action of A_5 or S_5 on 5 points, so this is an extremely primitive example, which is recorded in line 1 of Table 1. In (ii)–(iv) it is easy to check that $|\Omega| - 1$ is not divisible by $|F(H)|$, whence G is not extremely primitive by Lemma 2.2(iii). For example, in (iv) we have $G = G_0.2$ and $H = 2^6.O_6^+(2)$ (see [12]), whence $F(H) = Z_2^6$ but $|\Omega| - 1 = 3838184$ is not divisible by 64. □

9. Classical subgroups

The members of Aschbacher's $\mathcal{C}_8$ collection are the stabilizers of non-degenerate forms defined on the natural G_0-module V. For example, if $G_0 = \mathrm{PSL}_n(q)$ and n is even then we may define a non-degenerate symplectic form on V, which yields a $\mathcal{C}_8$-subgroup of type $\mathrm{Sp}_n(q)$. The various possibilities for G and H are described in [14, Table 4.8.A].

Proposition 9.1. *Suppose $G_0 = \mathrm{PSL}_n(q)$ and $H \in \mathcal{C}_8$ is of type $\mathrm{Sp}_n(q)$. Then G is not extremely primitive.*

Proof. Here $n = 2m$ is even and $m \geq 2$. Let V denote the natural G_0-module and let $\{e_1, f_1, \ldots, e_m, f_m\}$ be a standard symplectic basis for V with respect to a non-degenerate symplectic form β. If G contains graph automorphisms of G_0 then $Z(H) = Z_2$ is nontrivial, so G is not extremely primitive by Lemma 2.2(i). For the remainder we may assume otherwise. Set $H_0 = \mathrm{PSp}_n(q) \leqslant H \cap G_0$ and let

$$J = \begin{pmatrix} & I_m \\ -I_m & \end{pmatrix}$$

be the matrix representing β with respect to the basis $(e_1, \ldots, e_m, f_1, \ldots, f_m)$, so modulo scalars we have

$$H_0 = \{x \in \mathrm{SL}_n(q) \mid xJx^T = J\}.$$

In addition, we note that (34) holds for all $x \in G_0$.

Suppose $n \geq 6$. Set $V_1 = \langle e_1, f_1, e_2, f_2\rangle$, $V_2 = \langle e_3, f_3, \ldots, e_m, f_m\rangle$ and fix the basis ordering $(e_1, f_1, \ldots, e_m, f_m)$. Note that the H_0-stabilizer of the orthogonal decomposition $V = V_1 \perp V_2$ is a central product $H_1 \circ H_2$ with $H_1 \cong \mathrm{Sp}_4(q)$ and $H_2 \cong \mathrm{Sp}_{n-4}(q)$.

First assume q is even. Define

$$x = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & \\ 1 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 0 & 1 & \\ 0 & 1 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 1 & 1 & \\ \hline & & & & & & I_{n-6} \end{array}\right), \quad y = x^{-1}Jx^{-T}J^{-1} = \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & \\ \hline & & & & I_{n-4} \end{array}\right)$$

and note that $x \in G_0$. Now V_2 is the 1-eigenspace of y, so $C_{H_0}(y)$ fixes V_2 and thus $V_1 = V_2^{\perp}$ also. It follows that

$$L_0 \times H_2 = C_{H_0}(y) = H_0 \cap H_0^x < H_1 \times H_2 = (H_0)_{V_1,V_2},$$

where L_0 is a subgroup of H_1 of type $\mathrm{Sp}_2(q) \times \mathrm{Sp}_2(q)$ when $q \equiv 1 \pmod 3$, otherwise L_0 is of type $\mathrm{Sp}_2(q^2)$. The usual argument now implies that $H \cap H^x < H_{V_1,V_2} < H$ and thus G is not extremely primitive.

Next suppose q is odd, and continue to assume that $n \geq 6$. Here we define

$$x = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & \\ \hline & & & & I_{n-4} \end{array}\right), \qquad y = x^{-1}Jx^{-T}J^{-1} = \left(\begin{array}{cccc|c} 0 & 0 & -1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \\ \hline & & & & I_{n-4} \end{array}\right).$$

Once again, $x \in G_0$ and V_2 is the 1-eigenspace of y. We can now proceed as in the q even case.

Finally, let us assume $n = 4$. The cases with $q \leq 5$ are easily checked using Magma, so we may assume $q > 5$. Write $\mathbb{F}_q^* = \langle \omega \rangle$ and let $x \in G_0$ be the diagonal matrix $x = [\omega I_2, \omega^{-1} I_2]$ with respect to the basis (e_1, f_1, e_2, f_2), and set $y = x^{-1}Jx^{-T}J^{-1} = [\omega^{-2}I_2, \omega^2 I_2]$. Note that $\omega^2 \neq \omega^{-2}$ since $q > 5$. Set $U_1 = \langle e_1, f_1 \rangle$ and $U_2 = \langle e_2, f_2 \rangle$. Then

$$(H_0)_{U_1,U_2} = C_{H_0}(y) = H_0 \cap H_0^x < (H_0)_{\{U_1,U_2\}} < H_0$$

and thus $H \cap H^x < H_{\{U_1,U_2\}} < H$. We conclude that G is not extremely primitive. $\square$

Proposition 9.2. *Suppose $G_0 = \mathrm{PSL}_n(q)$ and $H \in \mathcal{C}_8$ is of type $O_n^{\varepsilon}(q)$ with q odd. Then G is not extremely primitive.*

Proof. Here $n \geq 3$ and q is odd (see [14, Proposition 4.8.4]). As in the proof of the previous proposition, we may assume G does not contain any graph automorphisms of G_0. Let Q be a non-degenerate quadratic form of type ε on V, with associated symmetric bilinear form β.

First assume $n \geq 5$. Fix a standard orthogonal basis $(e_1, f_1, e_2, f_2, \ldots)$ for V (with respect to Q), where $V_1 = \langle e_1, f_1, e_2, f_2 \rangle$ is a 4-space of plus type. The matrix J representing β is given in (35), so if we set $H_0 = \mathrm{PSO}_n^{\varepsilon}(q) \leqslant H \cap G_0$ then

$$H_0 = \{x \in \mathrm{SL}_n(q) \mid xJx^T = J\}$$

modulo scalars. Set $V_2 = V_1^{\perp}$ and note that $(H_0)_{V_1,V_2}$ is a central product $H_1 \circ H_2$, where H_1 is of type $O_4^+(q)$ and H_2 is of type $O_{n-4}^{\varepsilon}(q)$. Also note that (34) holds for all $x \in G_0$.

Take x and y as in the q odd case in the proof of Proposition 9.1, so $x \in G_0$ and V_2 is the 1-eigenspace of y. Then the same argument applies, giving

$$H \cap H^{x^{-1}} < H_{V_1,V_2} < H$$

so G is not extremely primitive.

To complete the proof, let us assume $n \leq 4$. In each of these cases, if $q \leq 5$ then the result can be checked via Magma so we will assume $q > 5$. Suppose $(n, \varepsilon) = (4, +)$. Fix a standard orthogonal basis (e_1, f_1, e_2, f_2) for V. Take $x = [\omega I_2, \omega^{-1} I_2] \in G_0$, where $\mathbb{F}_q^* = \langle \omega \rangle$, and $y = x^{-1}Jx^{-T}J^{-1} = [\omega^{-2}I_2, \omega^2 I_2]$, where

$$J = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

represents β. Set $U_1 = \langle e_1, f_1 \rangle$ and $U_2 = \langle e_2, f_2 \rangle$. Then in the usual manner we deduce that

$$H \cap H^x \leqslant H_{U_1,U_2} < H_{\{U_1,U_2\}} < H,$$

where $H_{\{U_1,U_2\}}$ is an imprimitive subgroup of type $O_2^+(q) \wr S_2$. The result follows.

Next suppose $(n, \varepsilon) = (4, -)$. Let $\{e_1, f_1, u, v\}$ be a standard orthogonal basis for V corresponding to a non-degenerate quadratic form Q of minus type (see [14, Proposition 2.5.3(ii)]). Let J be the matrix of β with respect to the specific basis ordering (e_1, f_1, u, v), so

$$J = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 2 & 1 \\ & & 1 & 2\lambda \end{pmatrix}$$

where $t^2 + t + \lambda \in \mathbb{F}_q[t]$ is an irreducible polynomial. Set

$$x = \begin{pmatrix} 1/2 & -1/2 & \\ 1 & 1 & \\ & & I_2 \end{pmatrix} \in G_0, \qquad y = x^{-1}Jx^{-T}J^{-1} = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & I_2 \end{pmatrix}.$$

Then $H_0 \cap H_0^x = C_{H_0}(y)$ is the set of matrices in H_0 of the form

$$\begin{pmatrix} a & b & \\ -b & a & \\ & & * \end{pmatrix}$$

with $ab = 0$ and $a^2 - b^2 = 1$. Note that there are exactly 4 possibilities for the ordered pair (a, b) when $q \equiv 1 \pmod 4$, and only 2 when $q \equiv 3 \pmod 4$. In particular, we deduce that $H_0 \cap H_0^x < (H_0)_{U_1,U_2} < H_0$, where $U_1 = \langle e_1, f_1\rangle$ and $U_2 = \langle u, v\rangle$. More generally, $H \cap H^x < H_{U_1,U_2} < H$ and thus G is not extremely primitive.

Finally, suppose $n = 3$. Let $\{e_1, f_1, d\}$ be a standard orthogonal basis for V, so

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents β in this basis. Set $U_1 = \langle e_1 + f_1 + d\rangle$ and $U_2 = U_1^{\perp}$, so $V = U_1 \perp U_2$ is an orthogonal decomposition of V into non-degenerate subspaces. Define

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in G_0, \qquad y = x^{-1}Jx^{-T}J^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $H_0 \cap H_0^x = C_{H_0}(y)$ is the set of matrices in H_0 of the form

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

and thus $H_0 \cap H_0^x < (H_0)_{U_1,U_2} < H_0$. In the usual manner, we conclude that $H \cap H^x < H_{U_1,U_2} < H$ and the result follows. □

Proposition 9.3. *Suppose $G_0 = \mathrm{PSL}_n(q)$ and $H \in \mathcal{C}_8$ is of type $\mathrm{U}_n(q_0)$ with $n \geq 3$ and $q = q_0^2$. Then G is not extremely primitive.*

Proof. Let $\{e_1, f_1, \ldots\}$ be a standard unitary basis for V with respect to a unitary form β (see [14, Proposition 2.3.2]). Let $\bar{G}$ be the ambient simple algebraic group $\mathrm{PSL}_n(K)$, where K is the algebraic closure of $\mathbb{F}_q$, and let σ be a Frobenius morphism of $\bar{G}$ such that $(\bar{G}_{\sigma^2})' = G_0$ and $(\bar{G}_{\sigma})' = H_0 = \mathrm{PSU}_n(q_0)$. Without loss of generality, we may assume $\sigma = \tau\phi$ is the standard graph–field automorphism of $\bar{G}$ with respect to the above basis, so τ is the inverse-transpose graph automorphism and ϕ is the involutory field automorphism defined by $\phi : (a_{ij}) \mapsto (a_{ij}^{q_0})$. Write $\mathbb{F}_q^* = \langle\omega\rangle$.

To begin with, let us assume $q_0 \geq 4$. Let $U = \langle e_1, f_1, d\rangle$ be a non-degenerate 3-dimensional subspace of V such that $\beta(e_1, d) = \beta(f_1, d) = 0$ and $\beta(d, d) = 1$. Also set $V_1 = \langle e_1, f_1\rangle$ and $V_2 = V_1^{\perp}$. Define

$$x = \left(\begin{array}{ccc|c} 0 & \omega & 0 & \\ -\omega^{-1} & 0 & 0 & \\ 0 & 0 & 1 & \\ \hline & & & I_{n-3} \end{array}\right) \in G_0, \qquad y = x^{-1}\sigma(x) = \left(\begin{array}{cc|c} \omega^{q_0+1} & 0 & \\ 0 & \omega^{-1-q_0} & \\ \hline & & I_{n-2} \end{array}\right)$$

with respect to the basis ordering $(e_1, f_1, d, \ldots)$. Applying Lemma 7.2 we deduce that

$$H_0 \cap H_0^x = C_{H_0}(y) \leqslant (H_0)_{V_1,V_2} < H_0.$$

Moreover, since $q_0 \geq 4$ we have $\omega^{q_0+1} \neq \omega^{-1-q_0}$, so $C_{H_0}(y)$ is a proper subgroup of $(H_0)_{V_1,V_2}$ and in the usual way we deduce that $H \cap H^x < H_{V_1,V_2} < H$.

A very similar argument applies when $q_0 \leq 3$. Indeed, if $q_0 = 3$ we set

$$x = \left(\begin{array}{ccc|c} 0 & \omega^2 & \omega & \\ 1 & 1 & 1 & \\ \omega^6 & \omega & \omega^2 & \\ \hline & & & I_{n-3} \end{array}\right) \in G_0, \qquad y = x^{-1}\sigma(x) = \left(\begin{array}{cc|c} 1 & \omega^2 & \\ \omega^6 & -1 & \\ \hline & & I_{n-2} \end{array}\right),$$

while if $q_0 = 2$ we define

$$x = \left(\begin{array}{ccc|c} 1 & \omega & \omega^2 & \\ 0 & 1 & 1 & \\ \omega & \omega & 1 & \\ \hline & & & I_{n-3} \end{array}\right) \in G_0, \qquad y = x^{-1}\sigma(x) = \left(\begin{array}{cc|c} 1 & \omega^2 & \\ \omega & 0 & \\ \hline & & I_{n-2} \end{array}\right)$$

(in terms of the specific basis $(e_1, f_1, d, \ldots)$). Taking V_1 and V_2 as before, we see that V_2 is the 1-eigenspace of y and once again we conclude that $H \cap H^x < H_{V_1,V_2} < H$. □

Proposition 9.4. *Suppose $G_0 = \mathrm{PSp}_n(q)'$ and $H \in \mathcal{C}_8$ is of type $O_n^{\varepsilon}(q)$ with q even. Then G is extremely primitive if and only if $q = 2$, and then G occurs in line 2 of Table 1.*

Proof. Here G_0 is isomorphic to the orthogonal group $\Omega_{n+1}(q)$. In the case $n = 4$, we may suppose that G does not contain a graph–field automorphism because otherwise G has no maximal subgroup of type $O_n^{\varepsilon}(q)$ (see [1, (14.1)]). The action of G on the cosets of H is permutation isomorphic to the action of $\Omega_{n+1}(q)$ on the set of non-degenerate hyperplanes T of type ε of the natural $(n+1)$-dimensional module V. The non-degenerate quadratic form Q on V preserved by G has a non-singular radical $\mathrm{Rad}(V) = \langle d \rangle$, and $T \cap \mathrm{Rad}(V) = 0$ for each such T. Note that this G-action is 2-transitive if and only if $q = 2$. Let β denote the corresponding symmetric bilinear form on V.

Let $H = G_U$ be the stabilizer of a hyperplane U of V of type ε. For all singular 1-spaces $\langle u \rangle$ of U, we shall construct $q-1$ non-degenerate hyperplanes W of V of type ε such that $(U \cap W)^{\perp} = \langle u, d \rangle$ and $W \neq U$. These $q-1$ hyperplanes constitute a block of imprimitivity for the action of $H = G_U$ on all hyperplanes.

We make use of the standard basis $\{e_1, \dots, e_m, f_1, \dots, f_m, d\}$ for V given in [14, Proposition 2.5.3(iii)], where $n = 2m$, $Q(e_i) = Q(f_i) = 0$, $Q(d) \neq 0$, $\beta(e_i, e_j) = \beta(f_i, f_j) = \beta(e_i, d) = \beta(f_i, d) = 0$ and $\beta(e_i, f_j) = \delta_{i,j}$ for all i, j. More precisely, we choose d so that $Q(d) = \lambda$ and the polynomial $t^2 + t + \lambda \in \mathbb{F}_q[t]$ is irreducible.

As G is primitive on the hyperplanes of type ε we may assume that

$$U = \begin{cases} \langle e_1, \dots, e_m, f_1, \dots, f_m \rangle & \text{if } \varepsilon = + \\ \langle e_1, \dots, e_{m-1}, f_1, \dots, f_{m-1}, d_1, d_2 \rangle & \text{if } \varepsilon = - \end{cases}$$

where, for $\varepsilon = -$, $\langle d_1, d_2, d \rangle = \langle e_m, f_m, d \rangle$ and $\langle d_1, d_2 \rangle$ is a non-degenerate 2-space of minus type. Note that in both cases we have $U = \langle e_1, f_1 \rangle \perp U_0$ with U_0 a non-degenerate space of dimension $n-2$ and type ε.

For any $g \in G \setminus H$, let $K = H^g$ and $W = U^g$, so K is the G-stabilizer of W. Then $U \cap W$ has codimension 2 in V and hence $(U \cap W)^{\perp}$ is a 2-dimensional space containing $\mathrm{Rad}(V)$. Moreover $U \cap W$ is a hyperplane of the non-degenerate space U, and hence $U \cap (U \cap W)^{\perp} = \langle v \rangle$ and $(U \cap W)^{\perp} = \langle v, d \rangle$ for some $v \in U$.

We claim that for all singular 1-spaces $\langle u \rangle$ of U, there are exactly $q-1$ non-degenerate hyperplanes W of V of type ε such that $W \neq U$ and $(U \cap W)^{\perp} = \langle u, d \rangle$.

To prove our claim, note that, as H is transitive (indeed primitive) on the singular 1-spaces of U, we may assume that $u = e_1$, so $u^{\perp} = \langle u \rangle \perp U_0 \perp \mathrm{Rad}(V)$ and $u^{\perp} \cap U = \langle u \rangle \perp U_0$. Note that we must have $u \in U \cap W$ as otherwise $u \in (\langle u \rangle \perp (U \cap W))^{\perp} = U^{\perp}$, contradicting the fact that $U^{\perp} = \mathrm{Rad}(V)$. Since $U \cap W \subseteq u^{\perp} \cap U$ for each subspace W of type ε associated with u, it follows that $U \cap W = \langle u \rangle \perp U_0$ for each such W. Thus each such W is of the form $W = \langle u, w, U_0 \rangle$ for some $w \in V$. Note that $w \notin u^{\perp}$ as otherwise $u \in W^{\perp}$ contradicting the fact that $W^{\perp} = \mathrm{Rad}(V)$. Thus, multiplying w by a scalar if necessary, we may assume that $w = f_1 + w'$ for some $w' \in u^{\perp}$. Next, by adding an element of U_0 to w' if necessary, we may further assume that $w = ae_1 + f_1 + bd$ for some scalars a, b. If $b = 0$ then $w \in U$ and $U = W$, which we do not want. Thus $b \neq 0$. Now $\beta(e_1, w) = 1$, and $Q(w) = Q(ae_1 + f_1 + bd) = a + b^2 Q(d)$. Hence, for a given (non-zero) value of b, there is a unique a such that $Q(w) = 0$, namely $a = b^2 Q(d)$, and for this a we have exhibited a basis showing that the space W is non-degenerate of type ε (namely $(e_1, \dots, e_m, w, f_2, \dots, f_m)$ if $\varepsilon = +$ and $(e_1, \dots, e_{m-1}, w, f_2, \dots, f_{m-1}, d_1, d_2)$ if $\varepsilon = -$; in this latter case, if $m = 2$ this reads (e_1, w, d_1, d_2)). Also $(U \cap W)^{\perp} = \langle u, d \rangle$ as required. Distinct values of b give distinct spaces $W(b) = (\langle u \rangle \perp U_0) \oplus \langle f_1 + bd \rangle$, so we have exactly $q-1$ non-degenerate $W(b)$ of type ε for the given singular 1-space $\langle u \rangle$. This proves our claim. Note also that $H_{\langle u \rangle}$ acts transitively on these $q-1$ subspaces $W(b)$.

Let $\mathcal{W}$ be the set of non-degenerate hyperplanes W so that $(U \cap W)^{\perp} = \langle t, d \rangle$ for some singular $t \in U$. Those spaces W for which $(U \cap W)^{\perp} = \langle u, d \rangle$ for a fixed singular $\langle u \rangle \subseteq U$ form a block of imprimitivity of size $q-1$ and, as noted above, $H_{\langle u \rangle}$ acts transitively on those $q-1$ hyperplanes . Thus $\mathcal{W}$ is an orbit of H and if $q > 2$ then the H-action on $\mathcal{W}$ is imprimitive. Therefore, the G-action is not extremely primitive. On the other hand if $q = 2$ then the H-action on $\mathcal{W}$ is equivalent to its (primitive) action on singular 1-spaces of U, and in this case the G-action is 2-primitive and hence is extremely primitive. This case is recorded in line 2 of Table 1. □

10. Almost simple irreducible subgroups

Recall that Aschbacher's main theorem on the subgroup structure of G (see [1]) states that if H is a maximal subgroup of G then either H belongs to one of eight *geometric* subgroup collections (which we label $\mathcal{C}_i$, for $1 \leq i \leq 8$), or H is almost simple and acts irreducibly on the natural G_0-module. We write $\mathcal{C}_9$ to denote this latter collection of maximal subgroups (note that Kleidman and Liebeck [14] use $\mathcal{S}$, rather than $\mathcal{C}_9$, to denote this collection). These subgroups also satisfy a number of additional properties (see [14, p. 3]), which are introduced to ensure that a $\mathcal{C}_9$-subgroup is not contained in one of the geometric $\mathcal{C}_i$ collections. We also note that a small additional family of *novelty* subgroups arises when $G_0 = \mathrm{PSp}_4(q)'$ (with q even) or $\mathrm{P}\Omega_8^+(q)$—we will deal with these extra cases in Section 11.

Lemma 10.1. *Let G be an almost simple primitive classical group with socle G_0 and point stabilizer $H \in \mathcal{C}_9$. Then one of the following holds:*

(i) $b(G) = 2$.
(ii) *The action of G is permutation isomorphic to a subspace action.*
(iii) *(G, H) is one of the cases listed in Table 3, where $H_0 = \mathrm{Soc}(H)$.*

Table 3
$H \in \mathcal{C}_9$, $b(G) > 2$.

	G_0	H_0	Conditions
1	$\Omega_7(q)$	$G_2(q)'$	
2	$\mathrm{PSp}_4(q)$	$\mathrm{Sz}(q)$	q even, $\log_2 q > 1$ odd
3	$\mathrm{PSL}_4(2)$	A_7	
4	$\mathrm{PSL}_3(4)$	A_6	
5	$\mathrm{PSL}_2(19)$	A_5	
6	$\mathrm{PSL}_2(11)$	A_5	
7	$\mathrm{PSL}_2(9)$	A_5	
8	$\mathrm{PSU}_6(2)$	$\mathrm{PSU}_4(3)$	
9		M_{22}	
10	$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_3(4)$	
11		A_7	
12	$\mathrm{PSU}_3(5)$	A_7	
13		A_6	
14		$\mathrm{PSL}_3(2)$	$G = G_0.2$
15	$\mathrm{PSU}_3(3)$	$\mathrm{PSL}_3(2)$	
16	$\mathrm{Sp}_8(2)$	A_{10}	
17	$\mathrm{Sp}_6(2)$	$\mathrm{PSU}_3(3)$	
18	$\Omega_{14}^+(2)$	A_{16}	$G = G_0.2$
19	$\Omega_{12}^-(2)$	A_{13}	
20	$\Omega_{10}^-(2)$	A_{12}	
21	$\mathrm{P}\Omega_8^+(3)$	$\Omega_8^+(2)$	
22	$\Omega_8^+(2)$	A_9	
23	$\Omega_7(3)$	$\mathrm{Sp}_6(2)$	
24		A_9	

Proof. See Section 10 of [5]. $\square$

In view of Lemma 2.1 and our earlier analysis of subspace actions in Section 3, it remains to deal with the list of explicit cases recorded in Table 3.

Lemma 10.2. *If $G_0 = \Omega_7(q)$ and $H_0 = G_2(q)$ with q odd, then G is not extremely primitive.*

Proof. We may view G_0 as a subgroup of an 8-dimensional orthogonal group $X = \Omega_8^+(q)$ such that G_0 acts irreducibly on the 8-dimensional orthogonal space V. By [12, Proposition 3.1.1(iv)], $N_{\mathrm{Aut}(X)}(H_0)$ contains a triality automorphism τ of X. Moreover, by [12, Proposition 3.1.1(vi)] (noting that G_0 is a K_1-*group* in Kleidman's terminology), $G_0 \cap G_0^\tau \cong G_2(q)$ is the stabilizer in G_0 of a non-singular 1-space $\langle v\rangle$ of V. Since $H_0 = H_0^\tau$ it follows that $G_0 \cap G_0^\tau = H_0 = (G_0)_{\langle v\rangle}$. Multiplying the quadratic form Q preserved by X by an appropriate scalar, if necessary, we may assume that $Q(v) = 1$. Thus the action of G_0 on Ω is equivalent to its action on the set of 1-dimensional non-singular subspaces of V.

This G_0-action was analysed in [16, Proposition 2] and it was shown there that there exists an H_0-orbit Δ of length q^6-1. Thus $|(H_0)_\delta| = |G_2(q)|/(q^6-1) = q^6(q^2-1)$ for $\delta \in \Delta$, and it follows from the list of maximal subgroups of $G_2(q)$ in [13, Theorem A] that the only maximal subgroups containing a Sylow p-subgroup of H_0 are parabolic subgroups. Hence $(H_0)_\delta$ is contained in a maximal parabolic subgroup M_0 of H_0. Now $|M_0| = q^6(q^2-1)(q-1)$, and so $(H_0)_\delta$ is a proper subgroup of M_0 and G_0 is not extremely primitive on Ω.

Finally, let us assume $G \neq G_0$. Since G leaves invariant the conjugacy class of stabilizers $G_2(q)$ in G_0, it follows that H induces only diagonal and field automorphisms on $H_0 = G_2(q)$. Therefore Lemma 2.3 applies: the stabilizer H_δ contains $(H_0)_\delta$ and hence contains a Sylow p-subgroup of H_0. Since $(H_0)_\delta$ is properly contained in a maximal parabolic subgroup of H_0, we conclude that H_δ is not maximal in H. $\square$

Lemma 10.3. *If $G_0 = \mathrm{PSp}_4(q)$ and $H_0 = \mathrm{Sz}(q)$ then G is not extremely primitive.*

Proof. Here q is even, $\log_2 q > 1$ is odd and $H \cap G_0 = H_0$. If G contains an involutory graph–field automorphism then $Z(H) \neq 1$, so we may assume $G = G_0.\langle\phi\rangle$ and $H = H_0.\langle\phi\rangle$, where ϕ is a field automorphism. According to [15, Table 1], there exists an element $x \in C_{G_0}(\phi)$ such that $|H_0 \cap H_0^x| = q^2$ (we can take x to be the root element labelled $x_{a+b}(1)$ in [15, Table 1]). Therefore $H_0 \cap H_0^x$ is properly contained in a maximal parabolic subgroup M_0 of H_0 and thus G_0 is not extremely primitive. If $G \neq G_0$ then Lemma 2.3 implies that $H \cap H^x$ is not maximal in H, so G is not extremely primitive in this case either. $\square$

Our main result for $\mathcal{C}_9$-subgroups is the following proposition. Here we adopt the standard $\mathbb{ATLAS}$ [8] notation for the conjugacy classes of involutions in G.

Proposition 10.4. *Let G be an almost simple primitive classical group with socle G_0 and point stabilizer $H \in \mathcal{C}_9$. Let H_0 denote the socle of H. Then G is extremely primitive if and only if (G, H) is one of the following:*

(i) *$G_0 = \mathrm{PSL}_4(2)$ and $H_0 = A_7$ (line 4 of Table 1).*
(ii) *$G_0 = \mathrm{PSL}_3(4)$, $H_0 = A_6$ (line 6 of Table 1) and one of the following holds:*
  (a) *$G = G_0.\langle a, b\rangle = G_0.2^2$ and $H = H_0.2^2$, where $a \in 2C$, $b \in 2D$.*
  (b) *$G = G_0.2 = G_0.\langle a\rangle$ and $H = \mathrm{M}_{10}$, where $a \in 2B$ is an involutory graph–field automorphism.*

Table 4
The $\mathcal{C}_{10}$ collection of novelties.

G_0	Type of H	Conditions
$\mathrm{PSp}_4(q)'$	$O_2^{\varepsilon}(q) \wr S_2$	$q > 2$
	$O_2^-(q^2).2$	
	$P_{1,2} = [q^4].\mathrm{GL}_1(q)^2$	
$\mathrm{P}\Omega_8^+(q)$	$\mathrm{GL}_3^{\varepsilon}(q) \times \mathrm{GL}_1^{\varepsilon}(q)$	$q \geq 3$ if $\varepsilon = +$
	$O_2^-(q^2) \times O_2^-(q^2)$	
	$G_2(q)$	
	$[2^9].\mathrm{SL}_3(2)$	$q = p > 2$
	$P_{1,3,4} = [q^{11}].\mathrm{GL}_2(q)\mathrm{GL}_1(q)^2$	

(c) *$G = G_0.2 = G_0.\langle a\rangle$ and $H = \mathrm{PGL}_2(9)$, where $a \in 2D$ is an involutory graph automorphism.*

(iii) *$G = \mathrm{PSL}_2(11)$ and $H = A_5$ (line 7 of Table 1).*

(iv) *$G_0 = \mathrm{PSL}_2(9)$, $H_0 = A_5$ and either $G = G_0$ or $G \cong S_6$ (line 2 of Table 1 with $(n, \varepsilon) = (4, -)$).*

(v) *$G_0 = \mathrm{PSU}_4(3)$, $H_0 = \mathrm{PSL}_3(4)$ (line 5 of Table 1) and one of the following holds:*

(a) *$G = G_0.\langle a, b\rangle = G_0.2^2$ and $H = H_0.2^2$, where $a \in 2B$ is a diagonal involution of type $[-1, I_3]$ and $b \in 2F$ is an involutory graph automorphism with centralizer of type $O_4^-(3)$.*

(b) *$G = G_0.2 = G_0.\langle a\rangle$ and $H = H_0.\langle c\rangle$, where $a \in 2F$ and c is an involutory graph or graph–field automorphism.*

Proof. In view of Lemmas 10.1–10.3 we may assume that (G, H) is one of the cases numbered 3–24 in Table 3. In each of these cases we use Magma [3] to verify the desired result. For example, in the cases 3,4,6,7 and 10 (corresponding respectively to the cases labelled (i)–(v) in the statement of the proposition) we can use the MaximalSubgroups and CosetAction commands to construct G as an explicit permutation group on the set of right cosets of H in G; it is then straightforward to confirm the above results. Each of these extremely primitive examples is recorded in Table 1. (Note that the case labelled 7 appears in line 2 of Table 1 as the case $G_0 = \mathrm{PSp}_4(2)'$ with H of type $O_4^-(2)$.)

In each of the remaining cases we claim that G is not extremely primitive. To see this we use Magma to construct both G and H as explicit permutation groups, and then by random search we quickly identify an element $x \in G$ such that $H \cap H^x$ is not a maximal subgroup of H. From a computational perspective, the most difficult case here is when $G = O_{14}^+(2)$ and $H = S_{16}$; here the natural G-module V is the fully deleted permutation module for G over $\mathbb{F}_2$. First we note that $G = \mathrm{PSO}_{14}^+(2)$, which Magma stores as a permutation group on 8255 points. Next we construct H. According to the Web-Atlas [21], we have

$$A_{16} = \langle a, b \mid a \in 3A,\ b \in 15F,\ |ab| = 14,\ |abb| = 63\rangle .$$

Now, if $a \in 3A$ in A_{16} then $\dim C_V(a) = 12$ and thus $|a^G| = 10924032$. Similarly, if $b \in 15F$ in A_{16} then $|b^G| = 150360513379815847 15284480$. (To deduce this, we use the fact that the associated embedding arises from the fully deleted permutation module for A_{16} over $\mathbb{F}_2$—see [14, p. 185].) By random search, it is easy to find elements a and b in G such that

$$|a| = 3,\ |b| = 15,\ |a^G| = 10924032,\ |b^G| = 15036051337981584715284480.$$

Next, we use random search once again to find G-conjugates $e = a^c$ and $f = b^d$ such that $|ef| = 14$, $|ef^2| = 63$ and $|\langle e, f\rangle| = |A_{16}|$. Then $A_{16} = \langle e, f\rangle$ and we can take $H = N_G(\langle e, f\rangle)$. Using Magma it is easy to identify the order of every maximal subgroup of H and we then use random search to find an element $x \in G$ such that $|H \cap H^x|$ is not equal to the order of such a subgroup. In this way we deduce that G is not extremely primitive. □

11. Novelty subgroups

In order to complete the proof of Theorem 1.1, it remains to deal with the small additional collection of so-called *novelty* subgroups which arise in one of the following special cases:

(i) $G_0 = \mathrm{PSp}_4(q)'$, $p = 2$ and G contains graph–field automorphisms;

(ii) $G_0 = \mathrm{P}\Omega_8^+(q)$ and G contains triality automorphisms.

By a novelty subgroup, we mean a maximal subgroup H of G such that $H \cap G_0$ is not maximal in G_0. The possibilities arising in case (i) were described by Aschbacher (see [1, Section 14]), while those in case (ii) were determined later by Kleidman (see [12, Section 4]). We record the various cases in Table 4, and we use $\mathcal{C}_{10}$ to denote this subgroup collection.

Lemma 11.1. *Let G be an almost simple primitive classical group with socle G_0 and point stabilizer $H \in \mathcal{C}_{10}$. Then either $b(G) = 2$, or (G, H) is one of the cases listed in Table 5.*

Proof. See [5, Proposition 11.1]. □

Table 5
$H \in \mathcal{C}_{10}$, $b(G) > 2$.

G_0	Type of H	Conditions
$\mathrm{PSp}_4(q)'$	$O_2^-(q^2).2$	$q = 2$
	$P_{1,2} = [q^4].\mathrm{GL}_1(q)^2$	
$\mathrm{P}\Omega_8^+(q)$	$\mathrm{GU}_3(q) \times \mathrm{GU}_1(q)$	$q = 2, G = G_0.S_3$
	$G_2(q)$	
	$P_{1,3,4} = [q^{11}].\mathrm{GL}_2(q)\mathrm{GL}_1(q)^2$	

Proposition 11.2. *Let G be an almost simple primitive classical group with socle G_0 and point stabilizer $H \in \mathcal{C}_{10}$. Then G is not extremely primitive.*

Proof. In view of Lemmas 2.1 and 11.1, we may assume (G, H) is one of the cases listed in Table 5. First assume $G_0 = \mathrm{PSp}_4(q)'$. Using Magma it is easy to check that if $q = 2$ and H is of type $O_2^-(q^2).2$ then G is not extremely primitive. If H is a parabolic subgroup of type $P_{1,2}$ then $|F(H)| = q^4$, but q^4 does not divide $|\Omega| - 1 = (q+1)^2(q^2+1) - 1$ and thus Lemma 2.2(ii) or (iii) applies.

Next let us turn to the cases in Table 5 with $G_0 = \mathrm{P}\Omega_8^+(q)$. In the first case, the socle of H is not a product of isomorphic simple groups, while $Z(H) \neq 1$ when H is of type $G_2(q)$. In both cases we conclude that G is not extremely primitive. Finally, if H is of type $P_{1,3,4}$ then $|F(H)| = q^{11}$ and it is easy to check that q^{11} does not divide $|\Omega| - 1$. □

This completes the proof of Theorem 1.1.

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